Boolean circuits and efficient addition

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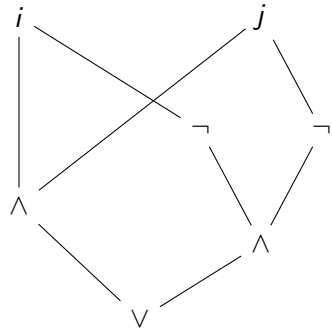
Parallel

- Boolean Circuits
- Lots of processors
- Each of them computes a single bit, using a Boolean operation
- Either uses inputs bits or result from other processors
- Connections are fixed

Boolean circuits

A first circuit



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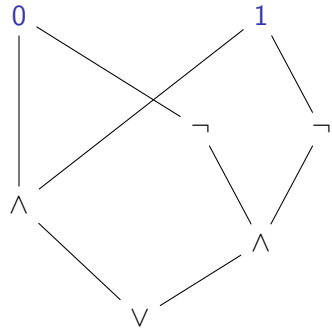

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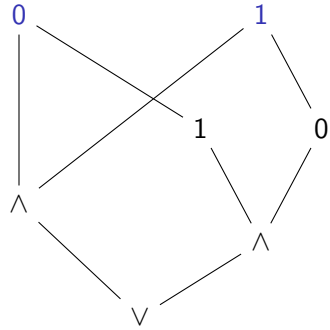

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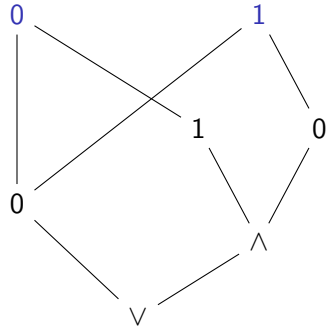

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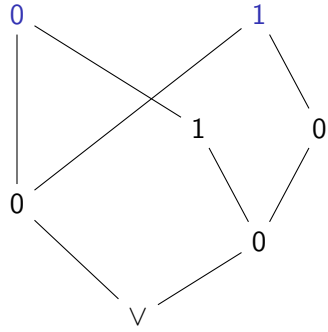

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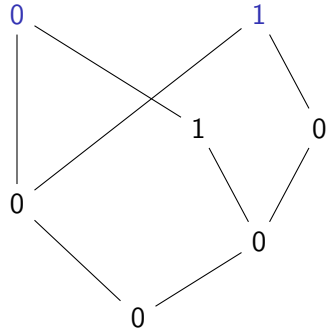

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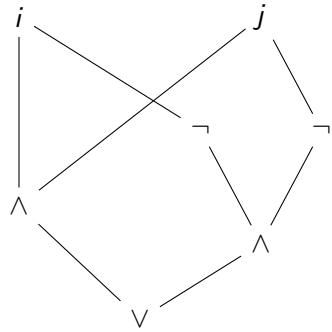

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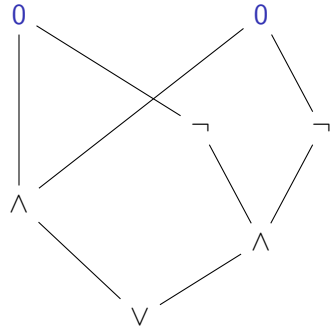

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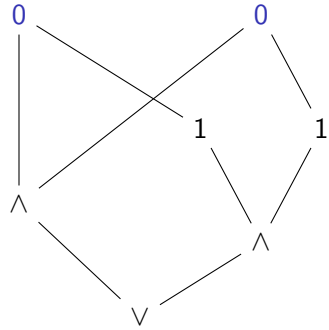

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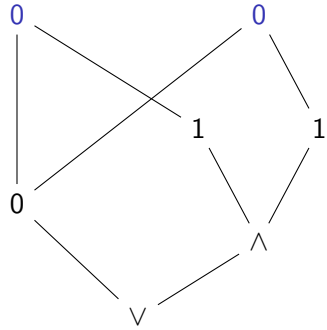

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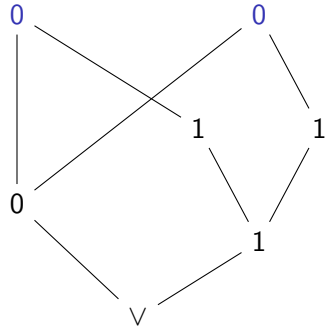

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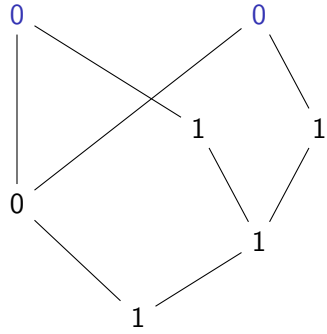

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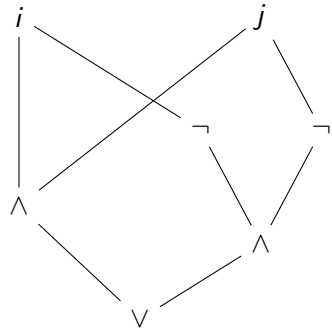

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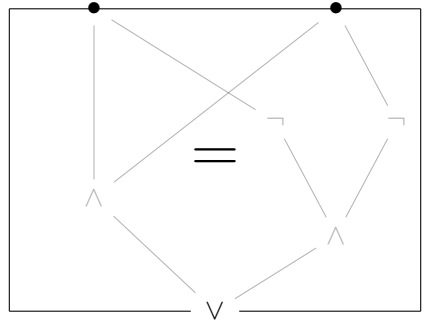

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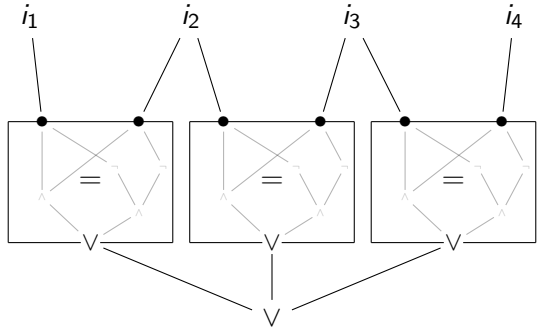

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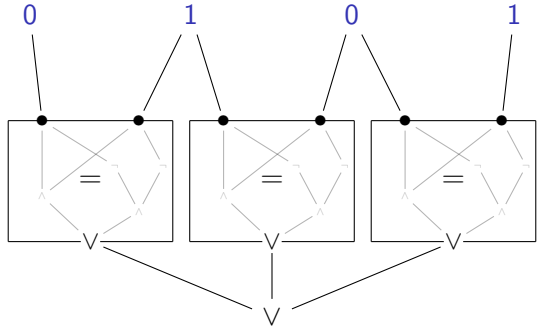

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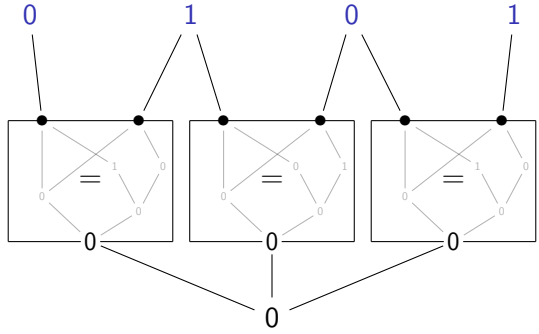

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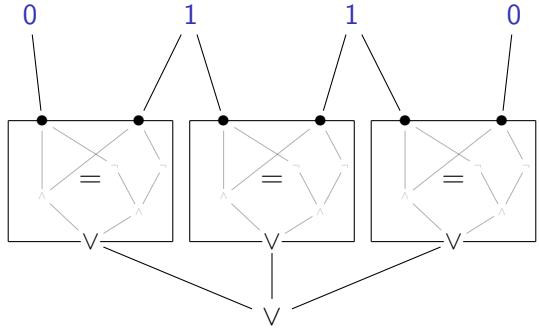

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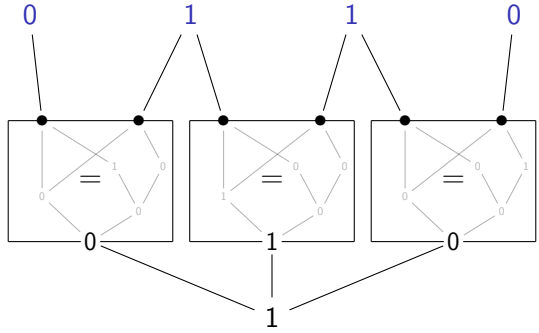

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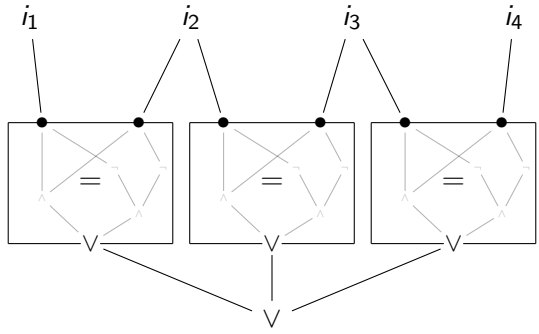

Computes $\Sigma^*(00 + 11)\Sigma^* \cap \Sigma^4$

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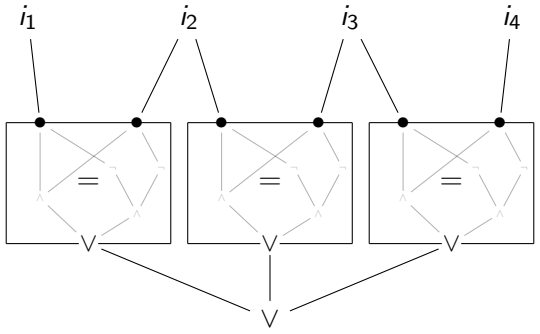

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size(n) = number of gates wire(n)= number of wires

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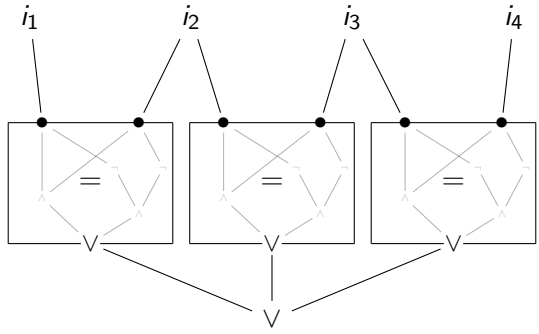

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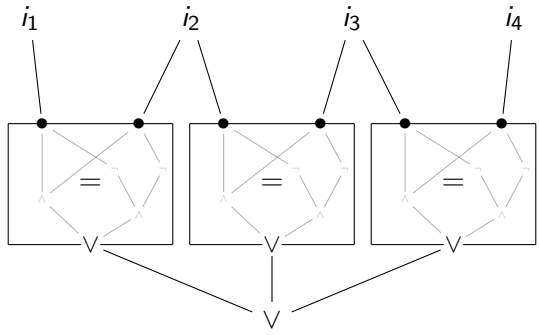

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$\text{size}(n) = 5 \cdot (n-1) + 1$

$\text{wire}(n) = 9 \cdot (n-1)$

$\text{depth}(n) = 4$

$\text{fan-in}(n) = n - 1$

Formal definitions

Definition (Syntax)

A circuit C over the variables $X = \{x_1, \cdots, x_n\}$ is a triple (G, λ, g_o) with

- a directed acyclic graph $G = (V, E)$
- a labelling of nodes $\lambda : V \to \{0, 1, \wedge, \vee, \neg\} \cup X$
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- $\lambda(g) \in \{0, 1\}$: $\lambda(g)$
- $\lambda(g) = x$ for $x \in V$: $\alpha(g)$
- $\lambda(g) = \neg$: 1 iff $v_\alpha(g') = 0$ for the unique g' with $(g', g) \in E$
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The ouput of the circuit is $v_\alpha(g_o)$.
The circuit computes a function $f_C : \{0, 1\}^n \to \{0, 1\}$ defined by $f(\alpha) = v_\alpha(g_o)$.

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Example

Denote by Acc_n the circuit with n inputs that always accepts (one "1" gate and no wires). Similarly, define Rej_n that always rejects.
Consider the family $\mathcal{C}$:

- $C_n = \text{Acc}_n$ if the n^{th} TM (in some order) always halts
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Can be avoided with uniformity: a procedure to compute the n^{th} circuit in the family.
$\rightarrow$ Not here
$\rightarrow$ See the lectures on "Complexity Theory"

Links with Turing Machines

Machines with advice

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A Turing machine with advice M is a TM with two inputs x and y, and an advice function $a : \mathbb{N} \to \{0,1\}^*$.
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For the converse, we admit the following for the moment.

Lemma

There is a poly-size circuit for every language in P.

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Definition (P/poly)

The complexity class P/poly is the class of languages accepted by a TM running in polynomial time with a polynomial size advice.

$\to$ A non-uniform version of P.

Theorem

L is in P/poly
$\Leftrightarrow$
L is computable by a poly-size circuit

Proof ($\Leftarrow$)

- Evaluating a poly-size circuit can be done in poly time.
- The circuit is the advice.

For the converse, we admit the following for the moment.

Lemma

There is a poly-size circuit for every language in P.

Proof ($\Rightarrow$)

- Take M a P/poly TM.
- Construct D_n the circuit for M with inputs x and y of sizes n and $|a(n)|$, by the lemma.
- Take C_n to be D_n with the inputs in y replaced by $a(n)$.

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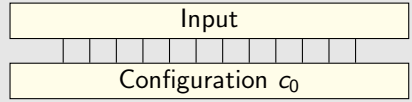

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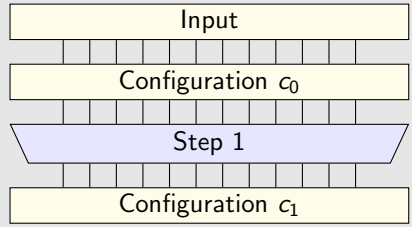

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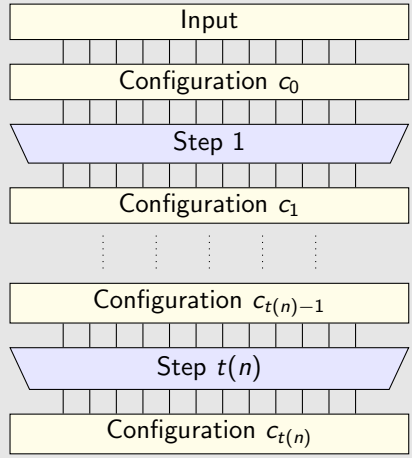

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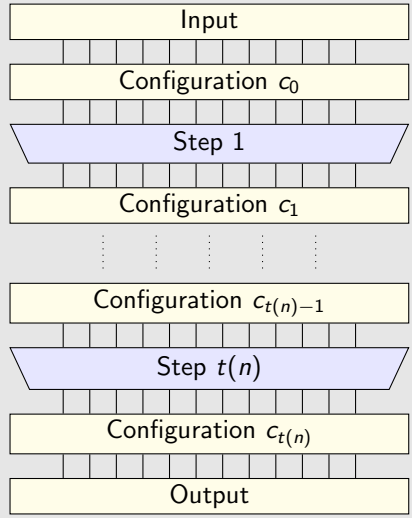

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We still do not know any explicit such function!

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We need to restrict circuits to study them.
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Proof

- We can remove the gates not linked to any wire.
- a $\vee$ gate of fan-in n can be transformed into a binary tree of size $2n$ and depth $\log(n)$.

Adding numbers

Adding two numbers

Problem (ADD)

- **Given:** Two n-bits numbers x and y
- **Output:** A $n+1$-bits number $z = x + y$

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- **Notation:** $\oplus$ denotes XOR
- **Idea:** Computes the successive carries:
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This is roughly the sequential algorithm.
$\rightarrow$ We can do better in parallel!

A better circuit for ADD

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ADD$\in$ AC^0

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Proof

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$\rightarrow$ Quadratic size (by factorizing the $(x_j \wedge y_j)$ and $(x_j \vee y_j)$)

$\rightarrow$ Constant depth

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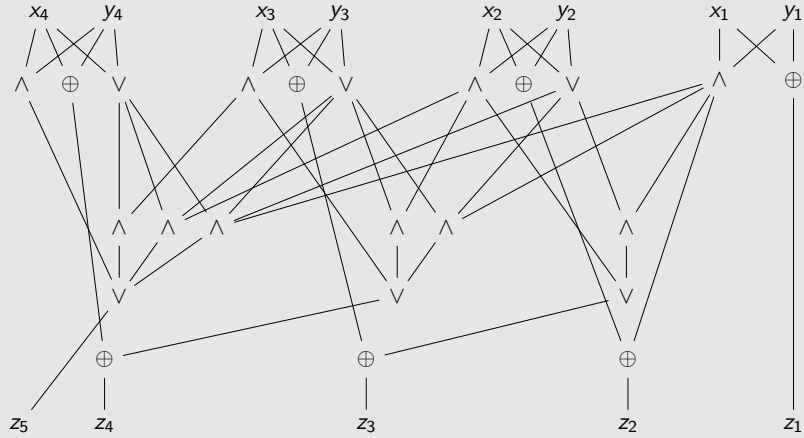

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The case of WLAC^0 will be the topic of the rest of this lecture.

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Intuition

- Adding the following numbers...

$$
\begin{aligned}
x_1 &= 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1 \\
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x_3 &= 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1
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- Adding the following numbers...

$$
\begin{aligned}
x_1 &= 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1 \\
x_2 &= 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \\
x_3 &= 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1 \\
s_4 &= 10
\end{aligned}
$$

Adding $\log(n)$ numbers

Lemma

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- ...reduces to adding the numbers:

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- For $1 \leq j \leq n$,
  $s_j = \sum_{i=1}^{\log(n)} j^{\text{th}}$ bit of x_i
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- **Last goal:** Add two $n + \log\log(n) + \cdots + \log^k(n)$ bits number where k = smallest integer such that the k-fold application of log on n is ≤ 2.

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- Small computation, for n big: $\log\log(n) + \cdots + \log^k(n) \leq \log(n)$
- We just have to add two $n + \log(n)$ numbers, that can be computed in AC^0 (depends on $\log(n)$ bits of the y_l)

The power of WLAC^0

Concentration of computation

We will study circuits from the properties of the underlying graph.

Definition (superconcentrator)

Let G be a DAG with n inputs $x_1, \ldots, x_n$ and n outputs $y_1, \ldots, y_n$. It is a superconcentrator if for all k and set of indices $i_1 < j_1 < \cdots < i_k < j_k$, there are k vertex disjoint paths from $\{x_{i_1}, \ldots, x_{i_k}\}$ to $\{y_{j_1}, \ldots, y_{j_k}\}$.

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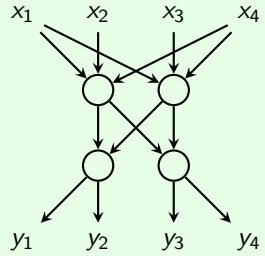
Example



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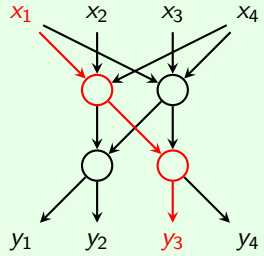

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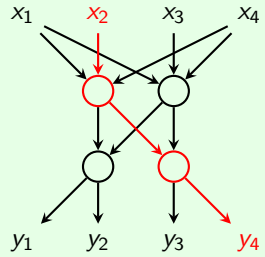

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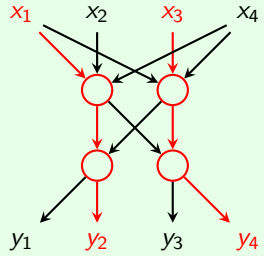

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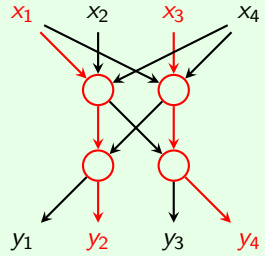
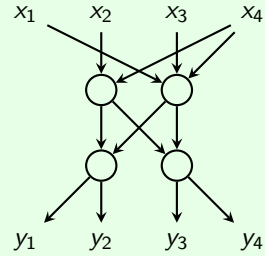


Counter-example



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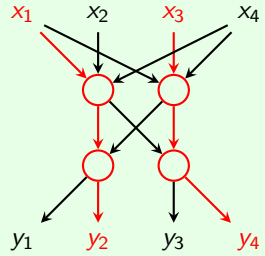

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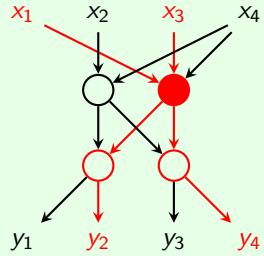

Efficient superconcentrators

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$\rightarrow$ To show that there are many vertex disjoint paths, we must show that there are no small cuts.

Back to ADD

Theorem

$\mathrm{ADD} \notin \mathrm{WLAC}^0$

Proof

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- Let C be a circuit for ADD with n inputs.

Back to ADD

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- C' is C with:
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  - x_i and y_i merged for $i \in I$

  $\rightarrow$ for $i \in I$, x_i and y_i must be set to the same value.
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- If there were a cut of size $< k$, there would be $< 2^k$ possible outcomes.

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Proof

- Let C be a circuit for ADD with n inputs.
- G is the graph with inputs x_i and y_i merged.
  $\rightarrow$ we want to show that it is a superconcentrator.
- Let $i_1 < j_1 < \cdots < i_k < j_k$.
  $\rightarrow$ sets I and J.
- C' is C with:
  - x_i set to 1 and y_i set to 0 for $i \notin I$
  - x_i and y_i merged for $i \in I$

  $\rightarrow$ for $i \in I$, x_i and y_i must be set to the same value.
  $\rightarrow$ k vertex disjoint paths in C' gives the same in G.

- **Claim:** the output z_{j_l} is 0 if x_{i_l} and y_{i_l} are both 1.
  the output z_{j_l} is 1 if x_{i_l} and y_{i_l} are both 0.
  $\rightarrow$ if $x_{i_l} = y_{i_l}$, what is on the right does not matter.
  $\rightarrow$ a carry is created and propagated if and only if both are 1.
- Thus there are 2^k possible outcomes for then outputs in J, depending on the inputs in I.
- If there were a cut of size $< k$, there would be $< 2^k$ possible outcomes.
- We conclude by Menger's theorem.

Recap

We have seen today:

	WLAC^0	AC^0	NC^1
ADD	✗	✓	✓
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Whether ADD can be done in constant depth with linearly many nodes is a major open problem.

Limitations of constant-depth circuits

Corentin Barloy Michael Walter Thomas Zeume

RUHR UNIVERSITÄT BOCHUM RUB

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The Parity language

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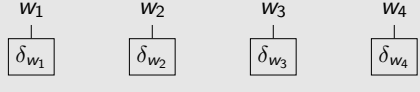

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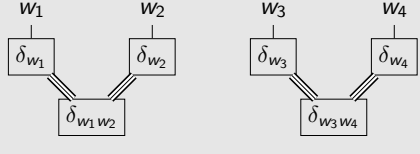

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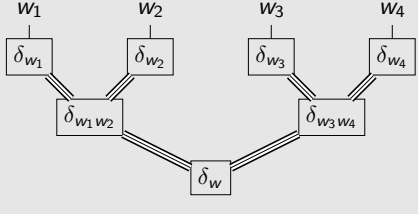

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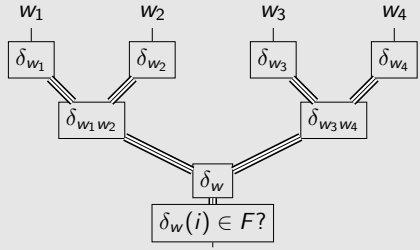

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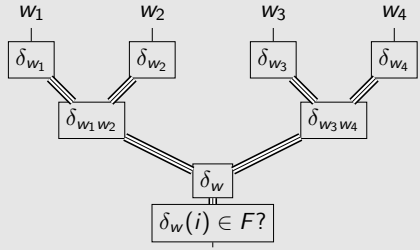

A binary tree of constant size circuits: logarithmic depth and linear size

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Both bounds works for CNF.

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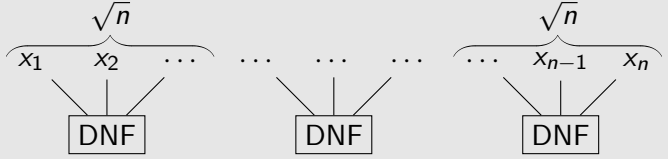

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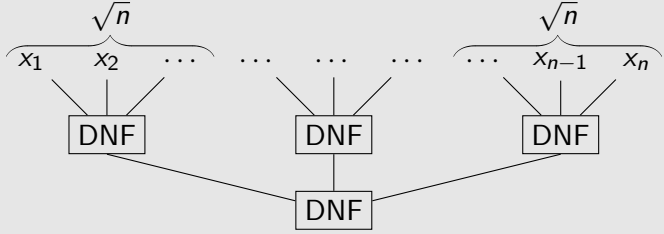

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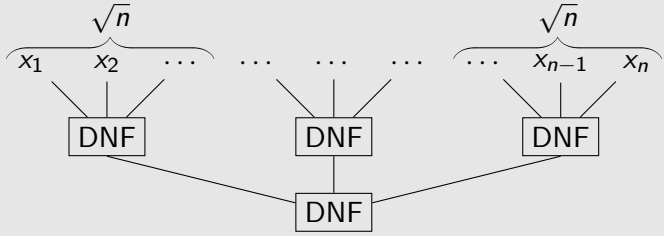
There is a depth-4 AC^0 circuit with $O(\sqrt{n}2^{\sqrt{n}})$ gates for Parity.

Proof

Idea: Compute Parity of small blocks, then compute Parity of the parities.



- So far: depth 6.

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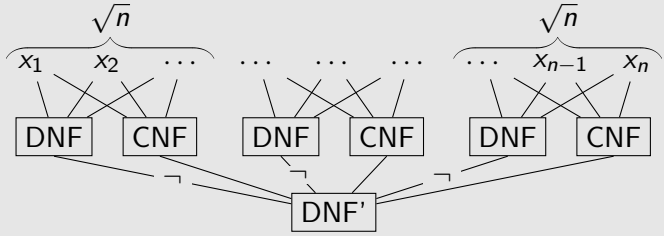

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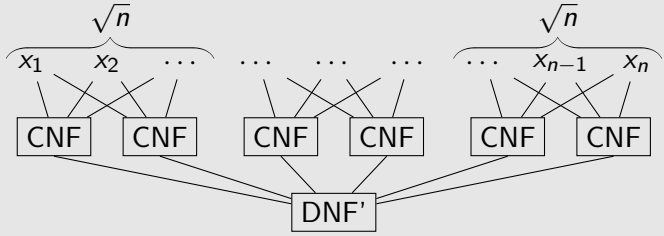

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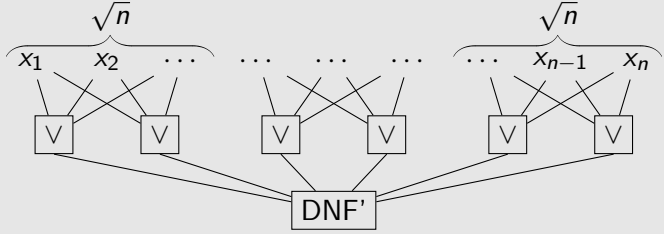

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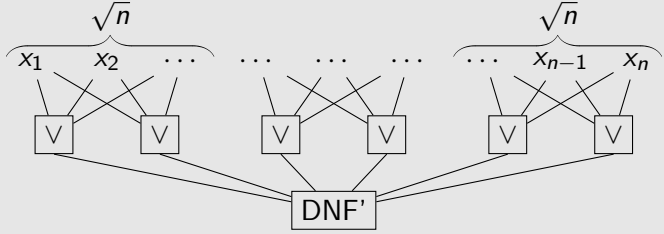

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If more depth is available, this can be improved even further.

Lemma (Håstad)

There is a depth-k AC^0 circuit with $O(n \cdot 2^{n^{\frac{1}{k-2}}})$ gates for Parity.

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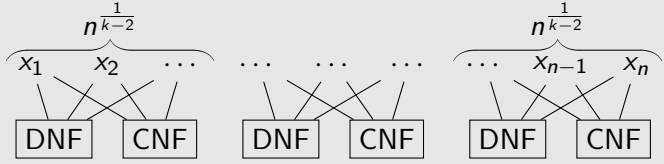

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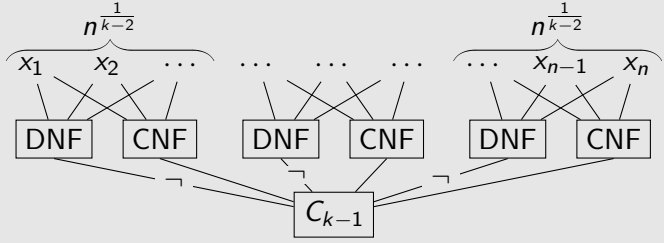

- C_{k-1} of depth k with a last $\wedge$ layer
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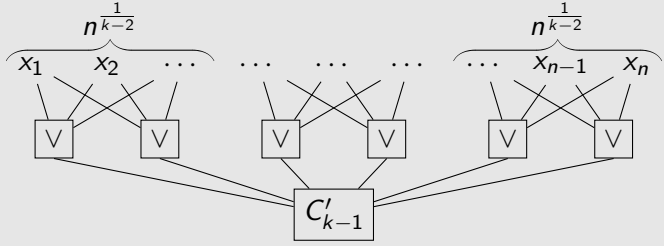

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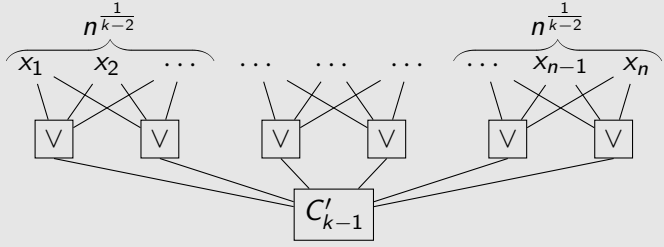

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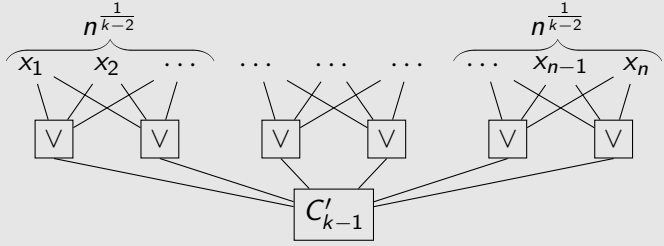

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- Total size $O(n \cdot 2^{n^{\frac{1}{k-2}}})$.

Proof 1: Switching lemma

A normal form

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Definition (Alternating circuits)

An alternating circuit of depth d is a circuit with

- variables or their negations as input
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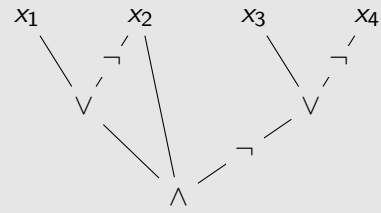

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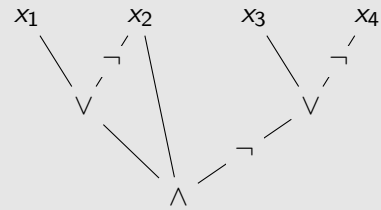

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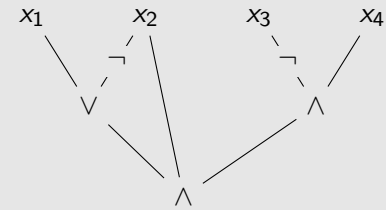

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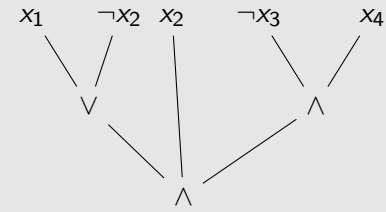

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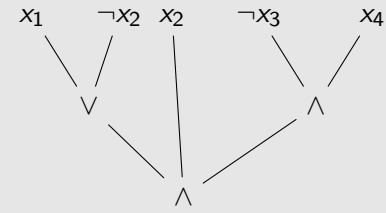

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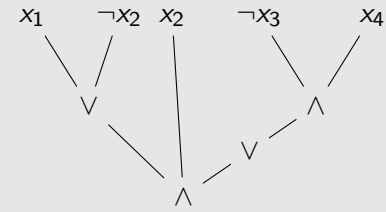

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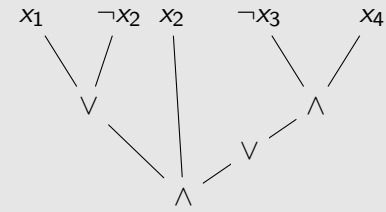

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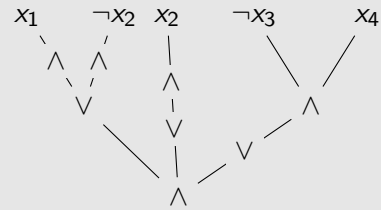

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Claim (Switching)

Let C be an alternating circuit of size ≥ 3. Replacing all CNFs at the leaves by equivalent DNFs reduces the depth by one.

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Claim (Switching)

Let C be an alternating circuit of size ≥ 3. Replacing all CNFs at the leaves by equivalent DNFs reduces the depth by one.

Proof

- By alternation, all parents of the CNFs are $\vee$-gates.
- After replacement, we can merge these $\vee$ with the DNF ones.

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Claim (Switching)

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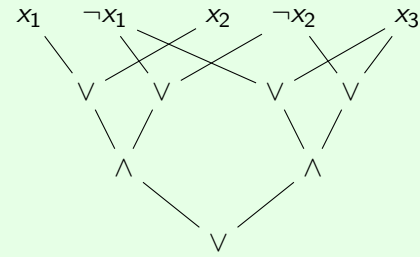
Example



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Claim (Switching)

Let C be an alternating circuit of size ≥ 3. Replacing all CNFs at the leaves by equivalent DNFs reduces the depth by one.

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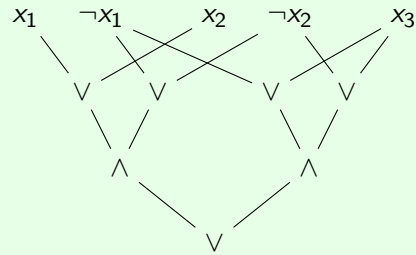

- $(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$
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Claim (Switching)

Let C be an alternating circuit of size ≥ 3. Replacing all CNFs at the leaves by equivalent DNFs reduces the depth by one.

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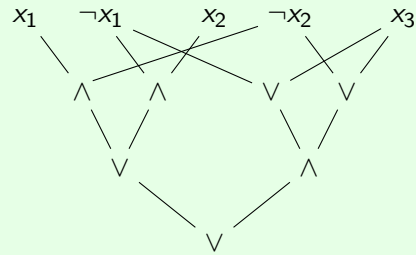

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Claim (Switching)

Let C be an alternating circuit of size ≥ 3. Replacing all CNFs at the leaves by equivalent DNFs reduces the depth by one.

Proof

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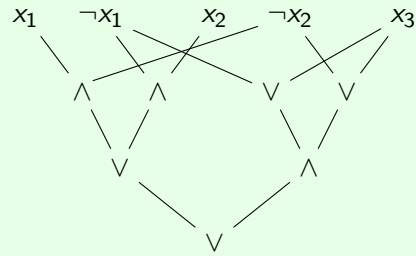

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Idea for lower bound for alternating circuits:

- We have shown that alternating of depth 2 cannot compute Parity.
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Let C be an alternating circuit of size ≥ 3. Replacing all CNFs at the leaves by equivalent DNFs reduces the depth by one.

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- By alternation, all parents of the CNFs are $\vee$-gates.
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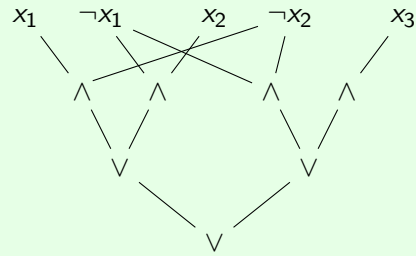

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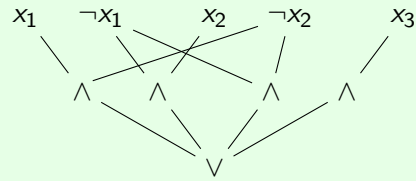

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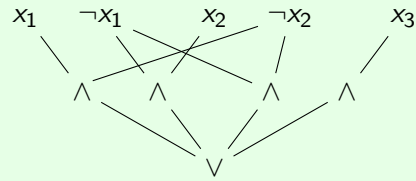

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$$\mathbb{P}_\rho(f_\rho \text{ has no } s\text{-DNF}) \leq (8pt)^s$$

where the probability is taken over all pn-restrictions.

$\to$ Admitted.

Theorem (Furst, Saxe and Sipser, and Håstad)

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- Key idea: Subfunctions of Parityare Parityitself or its negation.

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- By induction: $S^2 \leq 2^{\left(\frac{n}{16^2 m}\right)^{1/(k-1)}}$, which gives $\log(S) \leq c_k n^{\frac{1}{k}}$ for a constant c_k.

Proof 2: Polynomial approximation

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2nd **try:** polynomials over $\mathbb{F}_2$.

$$0 + 1 = 1 + 0 = 1$$
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→ circuits without sharing
→ Not helping
→ We want a field.

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$$0 + 1 = 1 + 0 = 1$$
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→ Parity is $\sum_{i=1}^{n} x_i$

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Polynomials are simpler objects than circuits.
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→ Seems that $\vee$ also needs high degree...

Last try: the approximation technique

Idea: Relax the notion of representation.
$\rightarrow$ we will approximate circuits with polynomials over $\mathbb{F}_3$.

Proof Idea

1) Approximate functions computed by AC^0 circuits by low-degree polynomials over $\mathbb{F}_3$.
2) Parity cannot be approximated by low-degree polynomials over $\mathbb{F}_3$.

Approximate means being correct on many inputs.

Definition

For $f(\overline{x})$ a function with n inputs and p a polynomial:
$\text{distance}(f,p) = |\{\overline{a} \in \{0,1\}^n \mid p(\overline{a}) \neq f(\overline{a})\}|$

Formalization of 1):

Lemma

Let C be a circuit of depth d and size M that computes a function f.
Then, for $1 \leq r \leq n$, there is a polynomial p of degree $\leq (2r)^d$ such that:

$$\text{distance}(f,p) \leq M \cdot 2^{n-r}.$$

$\rightarrow$ proved later.

Formalization of 2):

Lemma

There is a constant $c > 0$ such that every polynomial of degree $\leq \sqrt{n}$ satisfies:

$$\text{distance}(\text{Parity},p) \geq c \cdot 2^n.$$

$\rightarrow$ Admitted.

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Theorem (Razborov, Smolenski)

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- With $r = n^{\frac{1}{2d}}/2$: there is p of degree $\leq \sqrt{n}$ such that:
  $\text{distance}(\text{Parity}, p) \leq M \cdot 2^{n - n^{1/2d}/2}$.

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- Thus, $c \cdot 2^n \leq M \cdot 2^{n-n^{1/2d}/2}$.

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- $\neg$-gates do not introduce errors.
- for a $\vee$-gate, assume p_{h_k} is wrong for at most $M_k \cdot 2^{n-r}$ inputs, where M_k is the size of the subcircuit of h_k.
  Then p_g is wrong for at most $(M_1 + \cdots + M_k + 1) \cdot 2^{n-r}$ inputs.

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Claim

For $1 \leq r \leq n$, there is a polynomial $p_\vee$ of degree $\leq 2r$ such that distance$(\vee, p_\vee) \leq 2^{n-r}$

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- This proves the claim for any r.

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We have seen:

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> **Theorem**
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- Two different proofs:
  - Reducing the depth iteratively with random restrictions: switching lemma.
  - Approximate AC^0 circuits by low-degree polynomials.

Boolean circuits and regular languages

Corentin Barloy Michael Walter Thomas Zeume

RUHR UNIVERSITÄT BOCHUM RUB

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Definition (Completeness)

A language L is complete under projections for a class $\mathcal{C}$ if $L \in \mathcal{C}$ and there is a projection from every language in $\mathcal{C}$ to L.

Many separators and complete languages can be chosen regular.

Importance of regular languages

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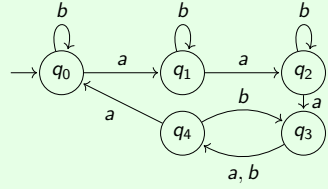

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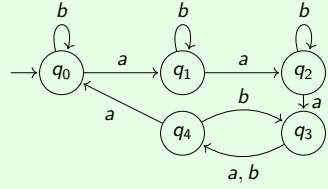

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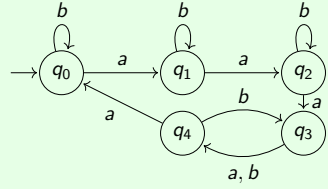

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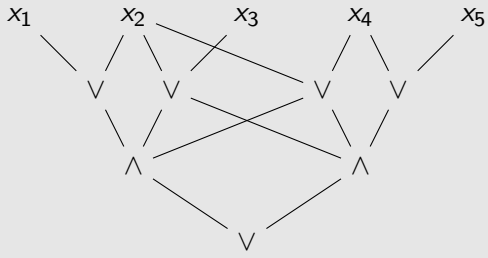

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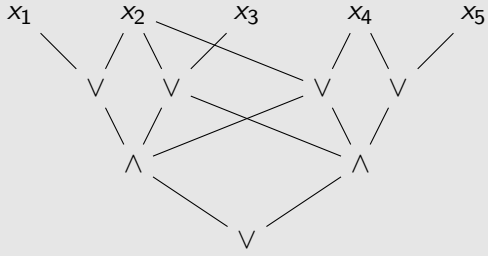

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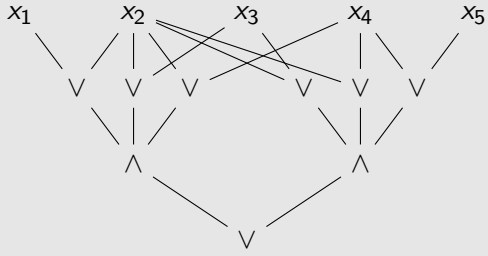

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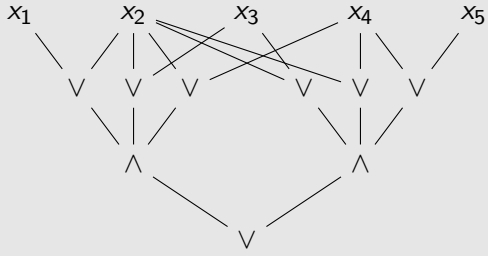

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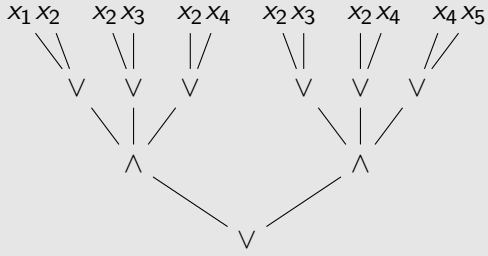

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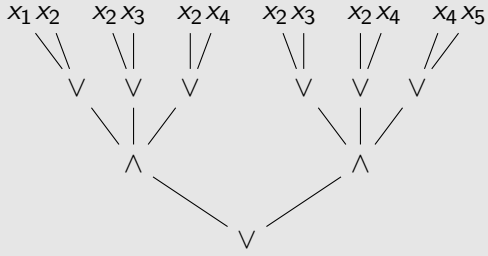

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- Add delimiters.

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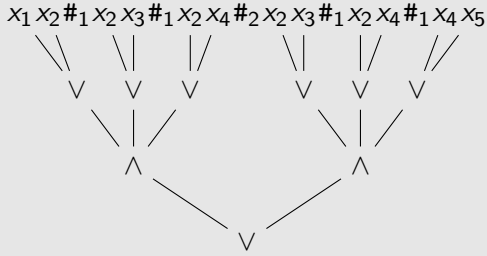

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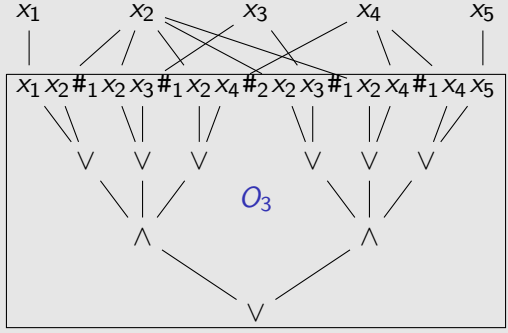

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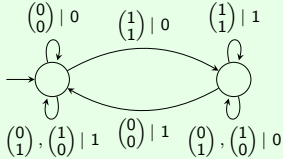

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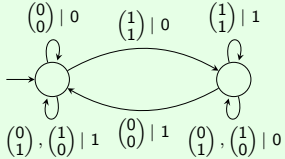

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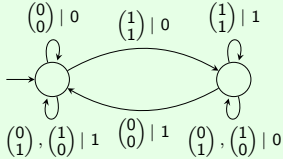

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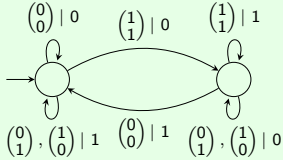

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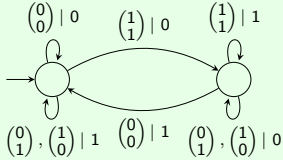

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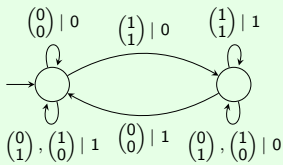

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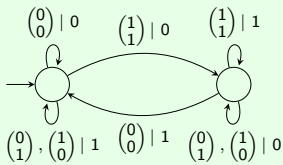

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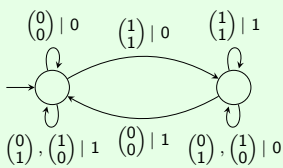

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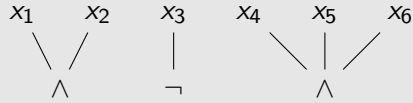

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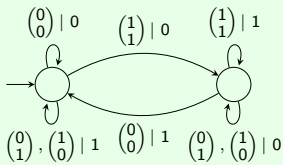

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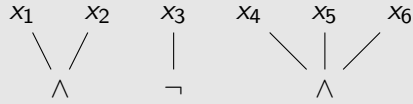

- Then we use:

	0	x_1	x_2	0	x_3	0	x_4	x_5	x_6
+	0	0	1	0	1	0	0	0	1
	$\downarrow$				$\downarrow$	$\downarrow$			

An algebraic toolbox

A new object

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A language L is recognized by a monoid M if there is a morphism $\mu : \Sigma^* \to M$ and $P \subseteq M$ such that $L = \mu^{-1}(P)$.

$\to$ Membership in L can be decided by looking only at information contained in M.

Claim
A language L is regular if and only if it is recognized by a finite monoid.

Proof
- $\mathcal{A} = (Q, \delta, i, F)$ DFA for L.
- δ_w is the extended transition function when reading w.
- Let M the set of functions $\delta_w : Q \to Q$ with the composition.
  $\to$ it is finite.
- Let $\mu : \Sigma^* \to M$ that maps w to δ_w.
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$\to$ This is the transition monoid of $\mathcal{A}$. It makes more structure visible.

A canonical monoid

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- We need: $u \sim_L v$ iff $\delta_u = \delta_v$.
- $\Leftarrow$: if $\delta_u = \delta_v$.
  - For x, y, assume $xuy \in L$.
  - Thus $\delta_{xuy}(i) \in F$.
  - Thus $\delta_{xvy}(i) \in F$.
  - Thus $xvy \in L$.
  - Hence $u \sim_L v$.
- $\Rightarrow$: if $u \sim_L v$.
  - Let $p \in Q$: by minimality, there is x such that $\delta_x(i) = p$.
  - For all y, we have
    $xuy \in L \Leftrightarrow xvy \in L$.
  - Thus, $\delta_{uy}(p) \in F \Leftrightarrow \delta_{vy}(p) \in F$.
  - Thus,
    $\delta_y(\delta_u(p)) \in F \Leftrightarrow \delta_y(\delta_v(p)) \in F$.

Another view on the syntactic monoic

Claim
The syntactic monoid is the transition monoid of the minimal automaton of L.

$\rightarrow$ M_L is finite and recognizes L.

Proof

- $\mathcal{A} = (Q, \delta, i, F)$ minimal DFA for L.
  $\rightarrow$ satisifes $p = q$ whenever
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  - Thus, $\delta_{uy}(p) \in F \Leftrightarrow \delta_{vy}(p) \in F$.
  - Thus,
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  - Thus,
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  - By minimality, $\delta_u(p) = \delta_v(p)$.
  - Hence, $\delta_u = \delta_v$.

$\rightarrow$ Also gives an algorithm to compute M_L.

An example

The syntactic monoid is generated by δ_a and δ_b: every element can be obtained are a product of these two.

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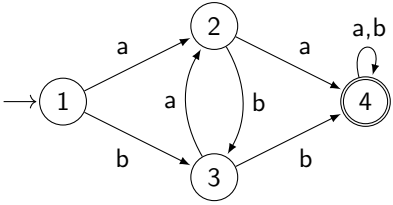
Computation of the syntactic monoid of $(a+b)^*(aa+bb)(a+b)^*$.

Minimal automaton



Syntactic monoid

	1	2	3	4

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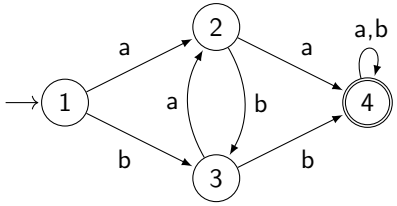

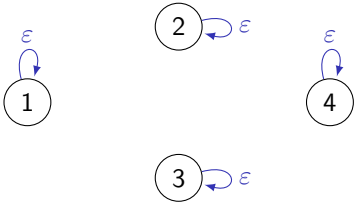

Syntactic monoid

	1	2	3	4
δ_ε	1	2	3	4

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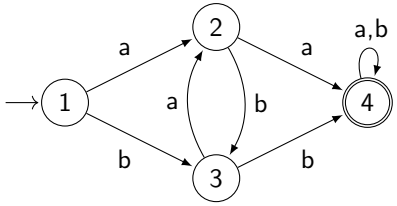

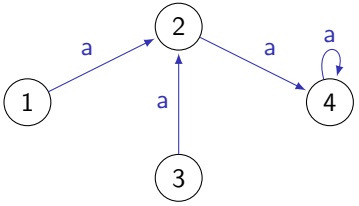

Syntactic monoid

	1	2	3	4
δ_ε	1	2	3	4
δ_a	2	4	2	4

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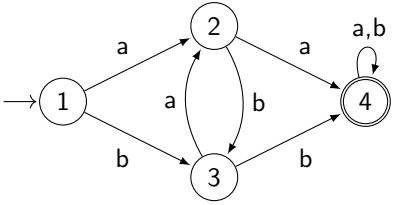

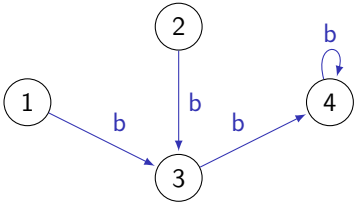

Syntactic monoid

	1	2	3	4
δ_ε	1	2	3	4
δ_a	2	4	2	4
δ_b	3	3	4	4

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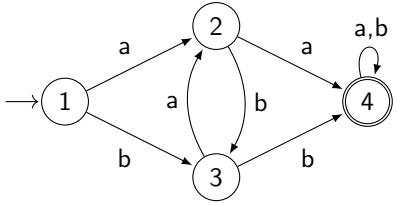

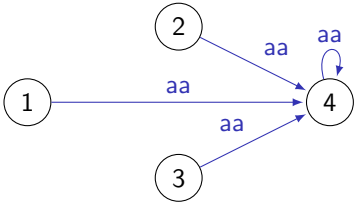

Syntactic monoid

	1	2	3	4
δ_ε	1	2	3	4
δ_a	2	4	2	4
δ_b	3	3	4	4
δ_{aa}	4	4	4	4

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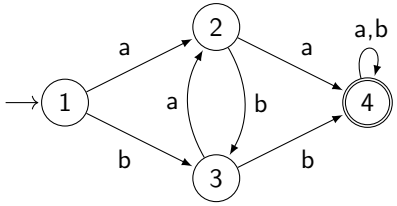

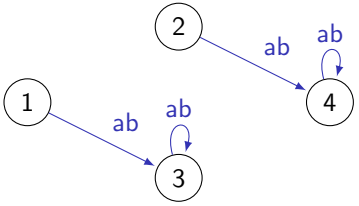

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	1	2	3	4
δ_ε	1	2	3	4
δ_a	2	4	2	4
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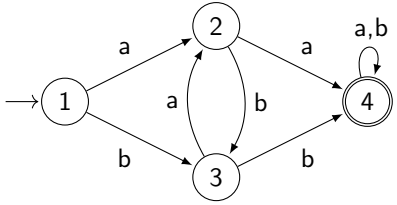

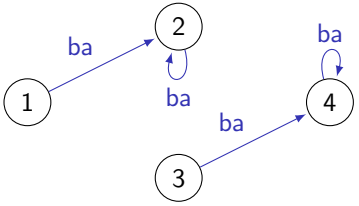

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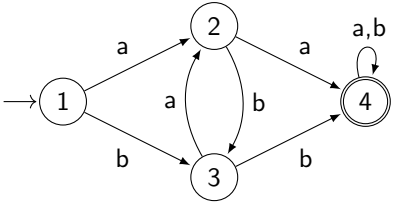

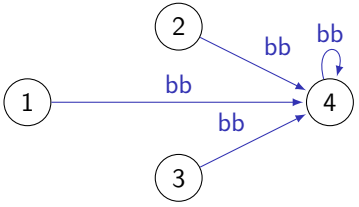

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	1	2	3	4
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$$\delta_b \cdot \delta_b \quad = \quad \delta_{aa}$$

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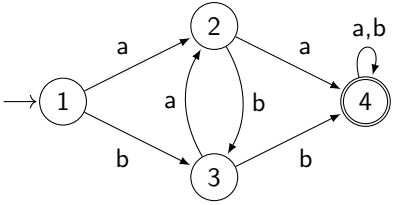

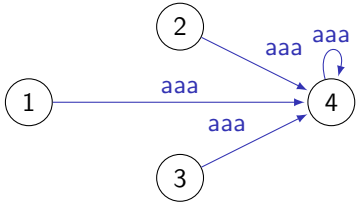

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δ_ε	1	2	3	4
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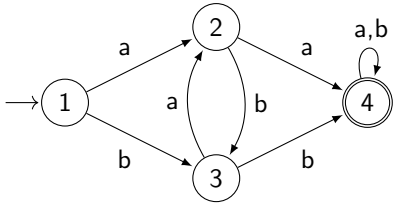

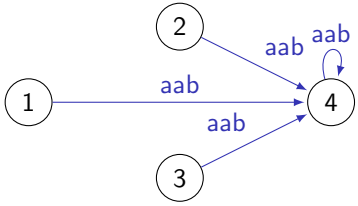

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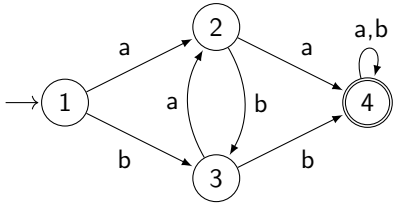

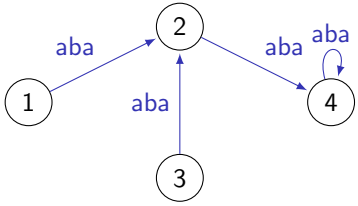

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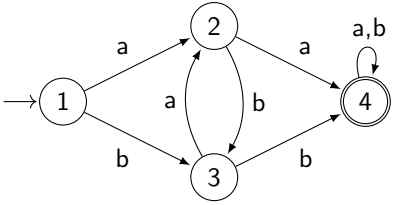

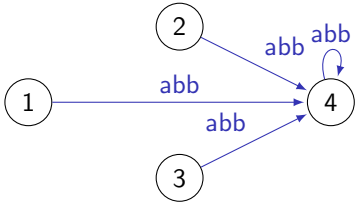

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$$\begin{aligned} \delta_b \cdot \delta_b &= \delta_{aa} \\ \delta_{aa} \cdot \delta_a &= \delta_{aa} \\ \delta_{aa} \cdot \delta_b &= \delta_{aa} \\ \delta_{ab} \cdot \delta_a &= \delta_a \\ \delta_{ab} \cdot \delta_b &= \delta_{aa} \end{aligned}$$

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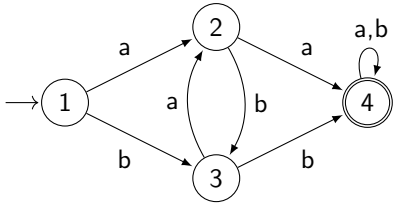

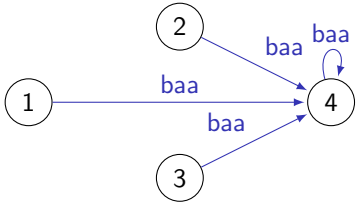

Syntactic monoid

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δ_ε	1	2	3	4
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$$\begin{aligned}
\delta_b \cdot \delta_b &= \delta_{aa} \\
\delta_{aa} \cdot \delta_a &= \delta_{aa} \\
\delta_{aa} \cdot \delta_b &= \delta_{aa} \\
\delta_{ab} \cdot \delta_a &= \delta_a \\
\delta_{ab} \cdot \delta_b &= \delta_{aa} \\
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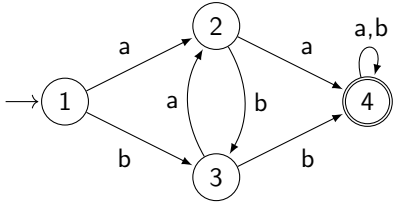

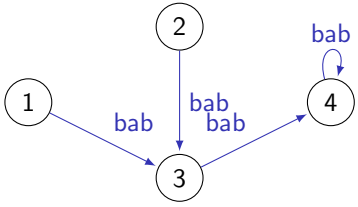

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	1	2	3	4
δ_ε	1	2	3	4
δ_a	2	4	2	4
δ_b	3	3	4	4
δ_{aa}	4	4	4	4
δ_{ab}	3	4	3	4
δ_{ba}	2	2	4	4

$$
\begin{aligned}
\delta_b \cdot \delta_b &= \delta_{aa} \\
\delta_{aa} \cdot \delta_a &= \delta_{aa} \\
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\delta_{ab} \cdot \delta_a &= \delta_a \\
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The regular languages of AC^0

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Claim

AC^0 languages are close under Boolean operations (union, intersection, complement).

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For the converse, we are stuck... we need algebra!

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$\Leftarrow$ Much more complicated: need a structure theory of monoids (Green's theory).

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Without a neutral letter, $(aa)^*$ is not star-free but is in AC^0.

Going further

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$\rightarrow$ Can be extended to depth-3 but not depth-4 so far.