Algebraic Characterizations of Classes of Regular Languages in DynFO

Corentin Barloy, Felix Tschirbs, Nils Vortmeier, Thomas Zeume

Incremental maintenance and DynFO

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→ We refine this with algebra!

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finite automaton $\approx$ finite monoid $(M, \cdot)$

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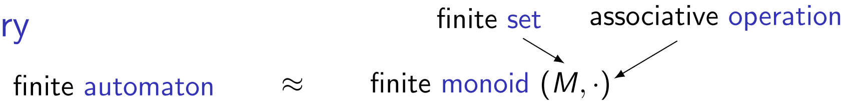

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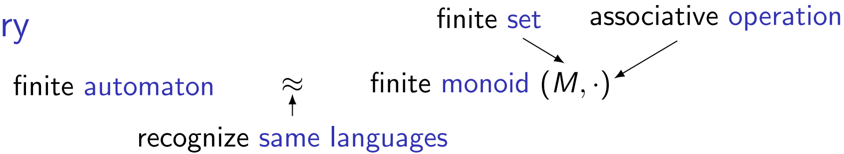

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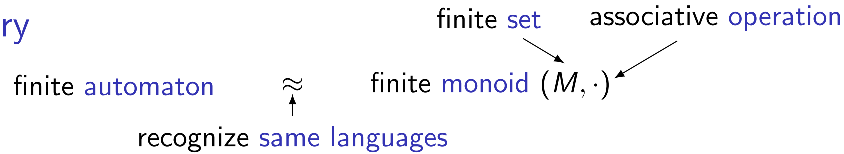

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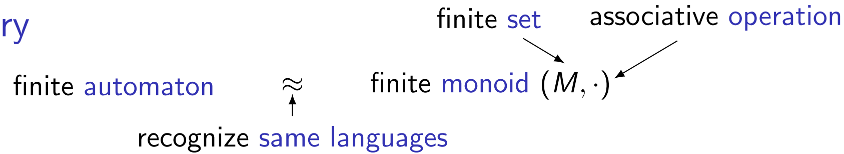

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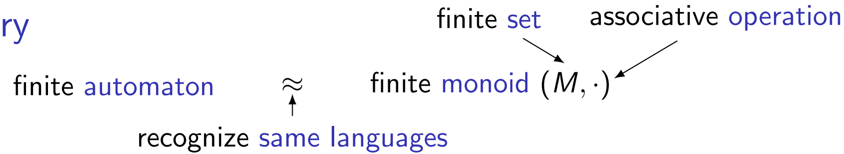

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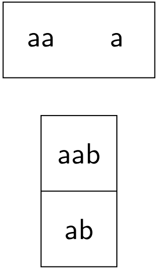
Previous proof of Reg $\subseteq$ DynFO: Maintain the evaluation of infixes in a monoid.

The regular languages of $\mathrm{UDyn}\Sigma_2$

Ideal structure

A division-based representation of the syntactic monoid of $(\mathtt{aa})^*\mathtt{b}(\mathtt{a}+\mathtt{b})^*$:



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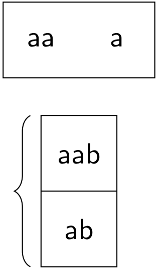

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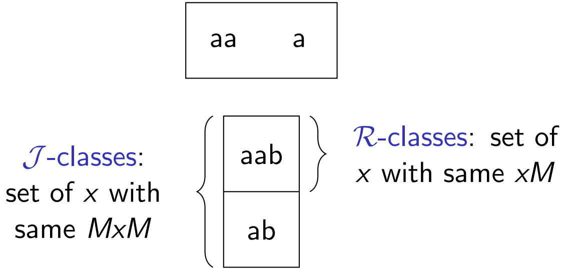

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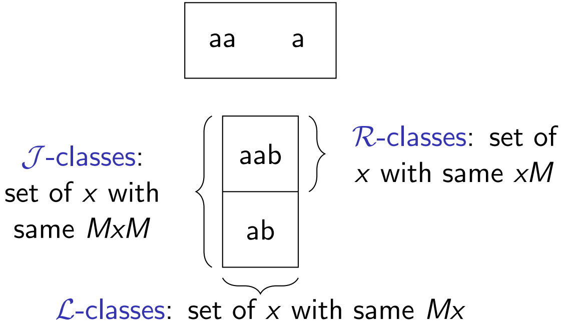

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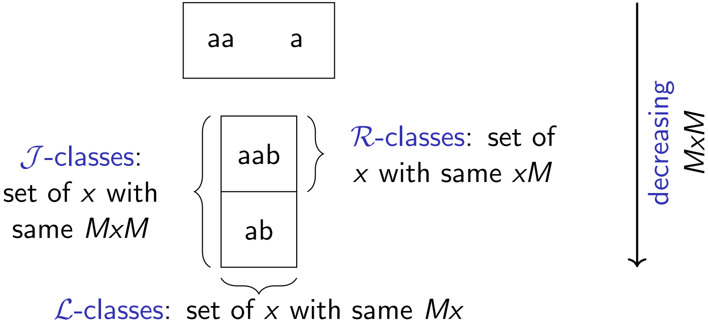

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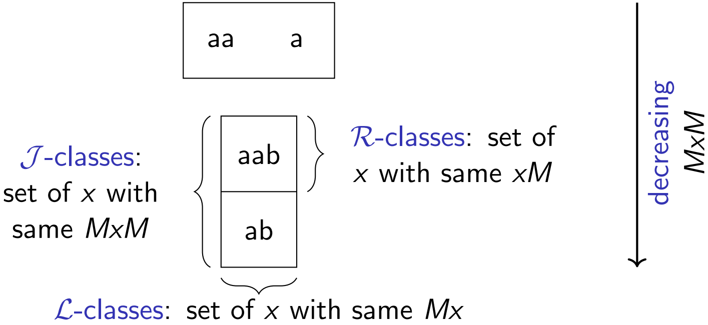

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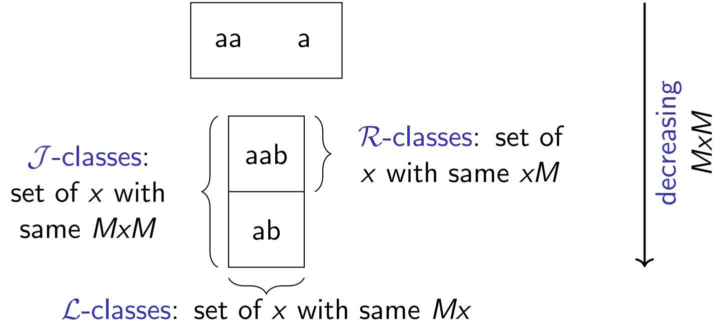

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→ Every word has a decomposition of the form:

$$\underbrace{\underbrace{\underbrace{w_1 \cdots\cdots w_{l_1}}_{w_1 \;\; \mathcal{J} \;\; x_1 \;\; <_{\mathcal{J}} \;\; x_1 w_{l_1+1}} \; w_{l_1+1} \cdots\cdots w_{l_2}}_{\substack{\mathcal{J} \\ x_2 \quad <_{\mathcal{J}} \quad x_2 w_{l_2+1}}} \; w_{l_2+1} \cdots\cdots w_n}_{\substack{\vdots \\ x_m}}$$

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Updates of $R_{J,x}$ at i: there is an index j such that $w[i,j]$ evaluates to x and $w[i,j+1]$ is $> J$.

The regular languages of UDynProp

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We used: maintain monoids ⇒ maintain all recognized languages
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→ This language in not in UDynProp [Schwentick, Zeume 2015]

The regular languages of UDynΣ_1^+

Positive varieties

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→ Membership of a language in a positive variety only depends on its syntactic ordered monoid

Wreath products

Sequential composition of automata $\mathcal{A}_1$ and $\mathcal{A}_2$: on input w, label w by the states it reaches in $\mathcal{A}_1$ and feed it to $\mathcal{A}_2$.

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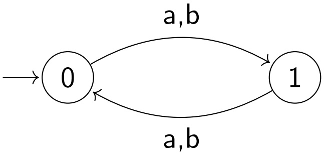

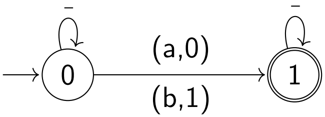

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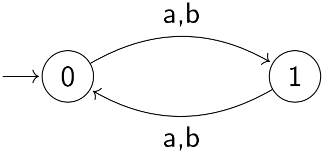

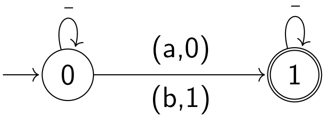

a	b	a	b	b	b

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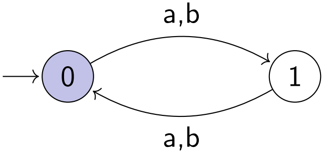

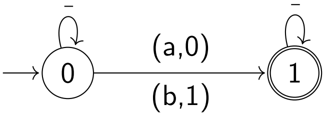

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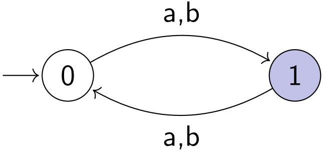

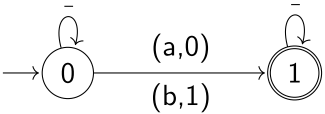

↓

a	b	a	b	b	b
1					

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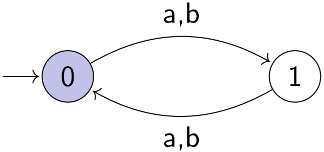

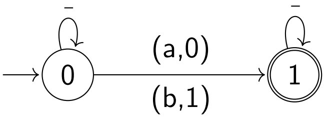

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a	b	a	b	b	b
1	0				

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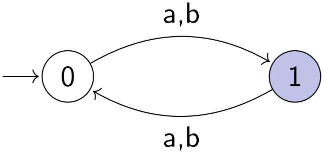

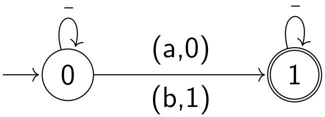

↓

a	b	a	b	b	b
1	0	1			

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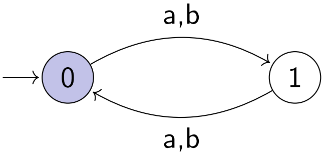

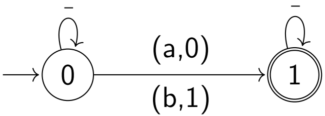

↓

a	b	a	b	b	b
1	0	1	0		

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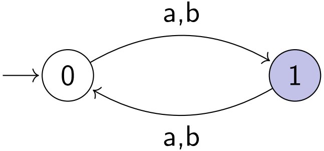

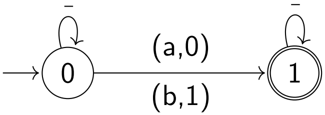

				↓	
a	b	a	b	b	b
1	0	1	0	1	

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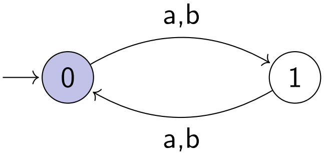

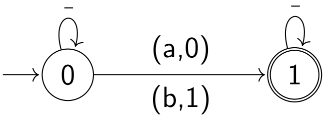

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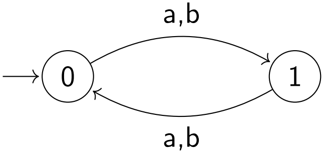

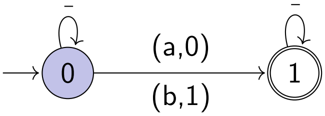

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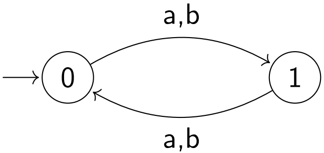

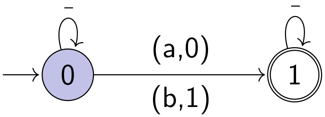

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a	b	a	b	b	b
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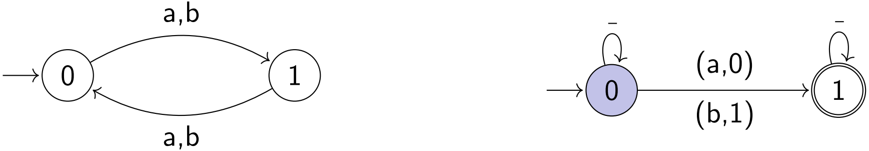

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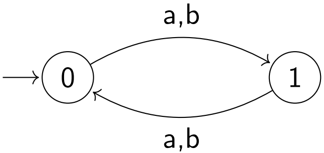

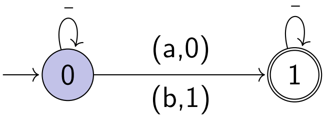

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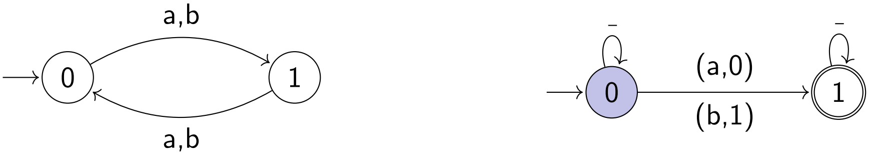

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a	b	a	b	b	b
1	0	1	0	1	0

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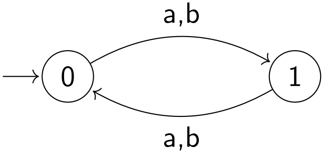

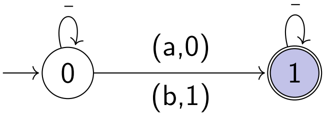

				↓	
a	b	a	b	b	b
1	0	1	0	1	0

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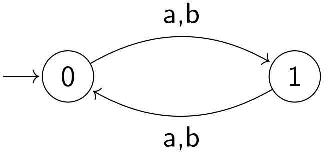

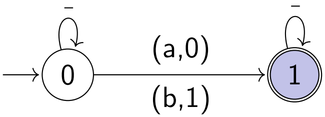

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a	b	a	b	b	b
1	0	1	0	1	0

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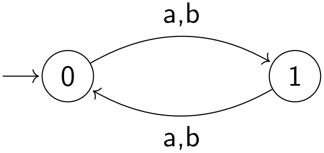

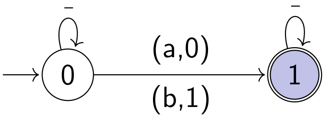

↓

a	b	a	b	b	b
1	0	1	0	1	0

Wreath product: Algebraic counterpart of the sequential composition, denoted $*$

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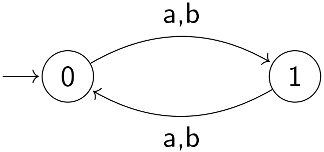

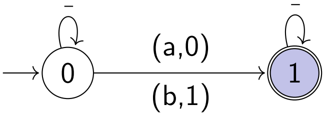

↓

a	b	a	b	b	b
1	0	1	0	1	0

Wreath product: Algebraic counterpart of the sequential composition, denoted $*$

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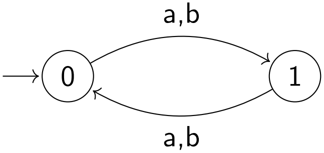

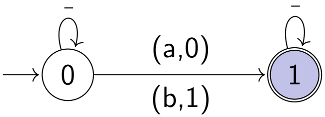

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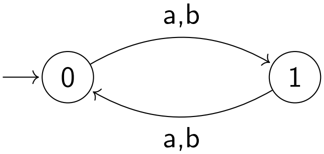

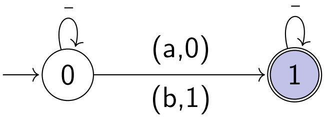

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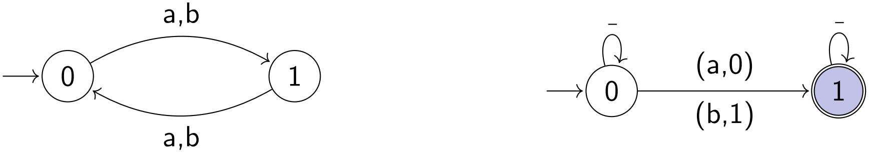

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Recap:

Dynamic class	UDynProp	$\mathrm{UDyn}\Sigma_1^+$	$\mathrm{UDyn}\Sigma_2$
Regular languages	$\mathbf{G}$	$\mathbf{J}^+ * \mathbf{G}$	Reg