A Robust Class of Linear Recurrence Sequences

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Several characterisations of LRS

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Theorem: The following classes 1 to 5 are equal.

1. Basic definition: $u_{n+k} = a_1 u_{n+k-1} + \cdots + a_k u_n$
2. Weighted automata
3. Linear cost register automata
4. Rational expressions
5. Rational power series

Running examples

1. Fibonacci: $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$.

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2. Identity: $u_{n+2} = 2u_{n+1} - u_n$ that is to say $u_n = n$.

(Unary) Weighted automata

Definition: A weighted automaton is a tuple $\mathcal{A} = (Q, M, I, F)$ where:

- Q is the finite set of states.
- M is the transition matrix in $\mathbb{Q}^{Q \times Q}$.
- I is the initial row vector in $\mathbb{Q}^{1 \times Q}$.
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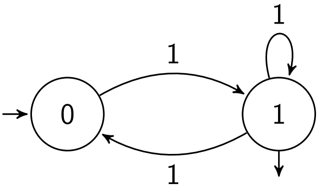
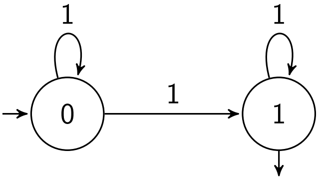
It recognizes the sequence $u_n = IM^nF$.

Fibonacci automaton



Identity automaton

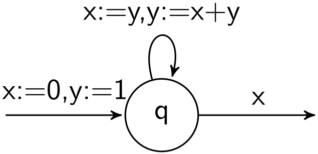


Linear cost-register automata

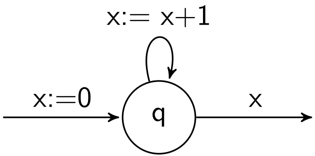
Definition: A cost-register automaton is a deterministic automaton with write-only registers. The key ingredients are:

- An initialisation of registers.
- A (linear) update of registers at each transition.
- Final expressions.

Fibonacci CRA



Identity CRA



Rational expressions

Given $\mathcal{C}$ a class of LRS and operators $\mathrm{op}_1, \ldots, \mathrm{op}_k$, the smallest class of LRS containing $\mathcal{C}$ and closed under those operators will be denoted by:

$$\mathrm{Rat}[\mathcal{C}, \mathrm{op}_1, \ldots, \mathrm{op}_k] \ .$$

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Kleene-Schützenberger

$$\mathrm{LRS} = \mathrm{Rat}[\mathrm{Fin}, +, \cdot, *]$$

Rational power series

We can associate to a LRS (u_n) the formal power serie $\sum_{\mathbb{N}} u_n X^n$.

Folklore

The sequences that are LRS are exactly those that can be written as a rational power serie,i.e., one of the form $\frac{P}{Q}$ with P and Q two polynomials.

The subclass of poly-rational sequences

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Theorem: The following classes 1 to 5 are equal.

1. Polynomially ambiguous weighted automata
2. Copyless cost register automata
3. Poly-rational expressions
4. LRS with roots of rationals as eigenvalues
5. Rational power series with roots of rationals as poles

Poly-rational expressions

- Arith is the set of sequences of the form $u_{n+1} = u_n + \lambda$.
- Geo is the set of sequences of the form $u_{n+1} = \lambda u_n$.
- shift is the operator that adds a value a in front of a sequence u.
- shuffle is the operator that interleaves sequences:

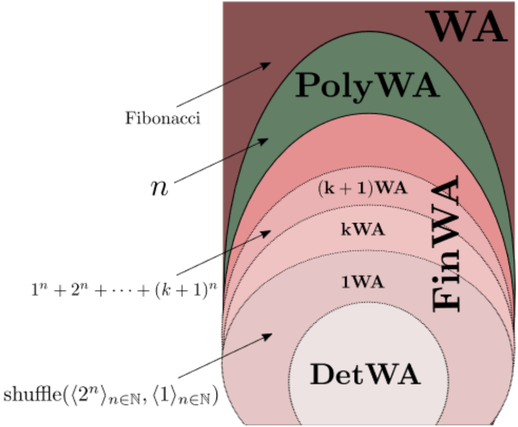
$$\text{shuffle}(u^1, \ldots, u^k) = (u^1_0, \ldots, u^k_0, u^1_1, \ldots, u^k_1, \ldots)\ .$$

Poly-rational sequences

$$\mathsf{PolyRat} = \mathsf{Rat}[\mathsf{Arith} \cup \mathsf{Geo}, +, \times, \mathsf{shift}, \mathsf{shuffle}] \quad .$$

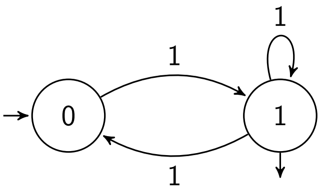
Ambiguity

The ambiguity function of $\mathcal{A}$ is the function that associates to $n \in \mathbb{N}$ the number of accepting runs for a^n.

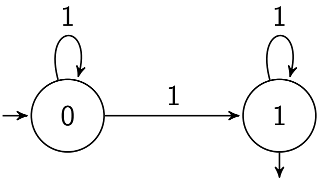


Examples

Exponentially ambiguous:



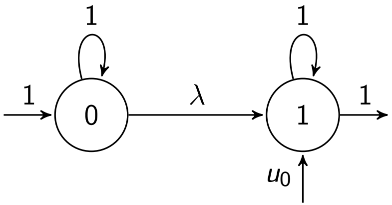
Polynomially ambiguous:



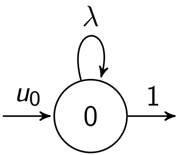
PolyRat $\subseteq$ PolyWA

Arith and Geo

$u_{n+1} = u_n + \lambda$:



$u_{n+1} = \lambda u_n$:



Closure under operators

- component-wise sum: disjoint union of automata.

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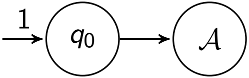

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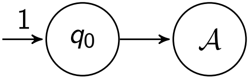
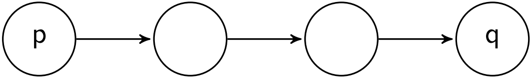


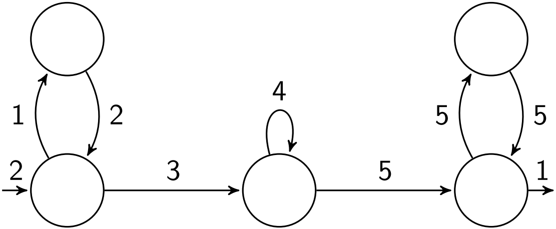
- shuffle: We add $k-1$ intermediate for each transition:



PolyWA $\subseteq$ PolyRat

Chained loop

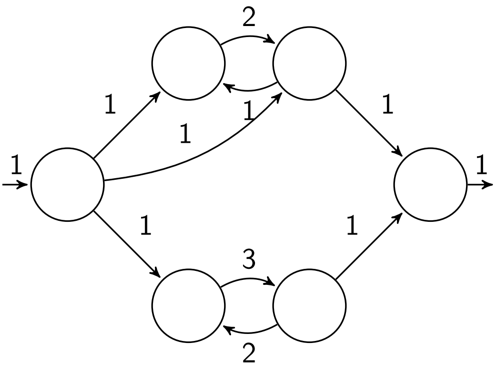
A chained loop is a simple path with disjoint loops along it (with only one initial and final state).



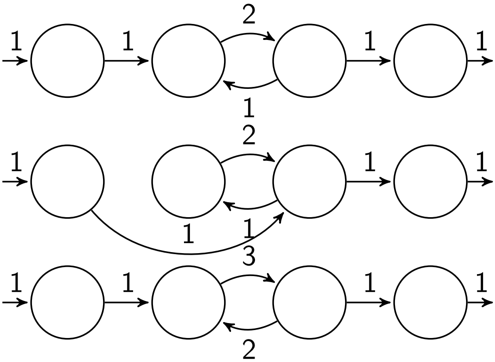
Decomposition

Theorem: Any polynomially ambiguous weighted automaton is equivalent to a union of chained loops.

Example



Example



Proof sketch

- The formal serie associated to any weighted automaton is of the form $\frac{P}{Q}$ where P and Q are polynomials and the roots of Q are roots of rationals.

$$Q(x) = 0 \Leftrightarrow \exists n, q, x = \sqrt[n]{q} \ .$$

Proof sketch

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$$Q(x) = 0 \Leftrightarrow \exists n, q, x = \sqrt[n]{q} \ .$$

- Such formal series define sequences in PolyRat.

Poles of PolyRat sequences

Folklore

If a LRS has a characteristic polynomial Q, then the induced serie is $\frac{P}{Q}$ for some polynomial P.

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If a LRS has a characteristic polynomial Q, then the induced serie is $\frac{P}{Q}$ for some polynomial P.

PolyRat sequences are exactly the LRS whose eigenvalues are roots of rationals.

Copyless cost-register automata

A cost-register automaton is copyless if every update uses each register at most once.

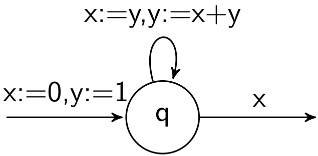
$$\begin{cases} x := x + 5y \\ y := 2z \\ z := 1 \end{cases}$$

$$\begin{cases} x := x \\ y := x + y \end{cases}$$

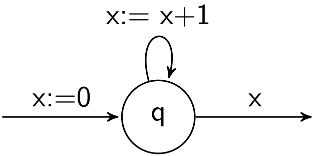
$$\begin{cases} x := x + y \\ y := 1 \end{cases}$$

Examples

Non copyless CRA:

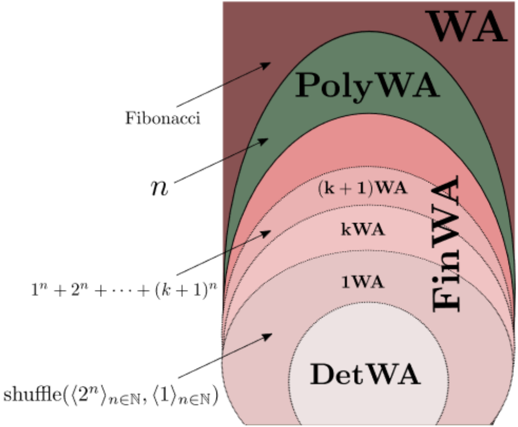


Copyless CRA:



$$\text{PolyRat} = \text{CCRA}$$

Strictness of the ambiguity hierarchy



Complexity

- The Skolem problem is decidable for our class [Rebiha14].
- The Skolem problem is NP-hard for our class [Akshay17].
- It is open if it is in NP.

Thanks!

Sources

1. Rachid Rebiha, Arnaldo Vieira Moura, and Nadir Matringe. On the termination of linear and affine programs over the integers.CoRR, abs/1409.4230, 2014.
2. S. Akshay, Nikhil Balaji, and Nikhil Vyas. Complexity of restricted variants of skolem and related problems. In 42nd International Symposium on Mathematical Foundations of Computer Science, MFCS 2017, August 21-25, 2017 - Aalborg, Denmark, pages 78:1–78:14,2017.