A Robust Class of Linear Recurrence Sequences

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Highlights of Logic, Games and Automata 2019 (Warsaw)

Several characterisations of LRS

Theorem: The following classes 1 to 5 are equal.

1. Basic definition
2. Weighted automata
3. Linear cost register automata
4. Rational expressions
5. Rational power series

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$$u_{n+k} = a_1 u_{n+k-1} + \ldots + a_k u_n$$

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$$u_{n+k} = a_1 u_{n+k-1} + \ldots + a_k u_n$$

$$Q(x) = x^k - a_1 x^{k-1} - \ldots - a_{k-1} x - a_k$$

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Fibonacci:

$$< 0, 1, 1, 2, 3, 5, \cdots >$$

$$F_{n+2} = F_{n+1} + F_n$$

$$Q(x) = x^2 - x - 1$$

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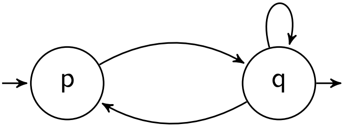

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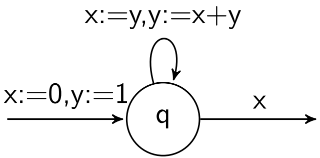

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$$\text{LRS} = \text{Rat}[\text{Fin}, +, \cdot, *]$$

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$$\sum_{i \in \mathbb{N}} u_i X^i = \frac{P}{Q}$$

with P and Q two polynomials.

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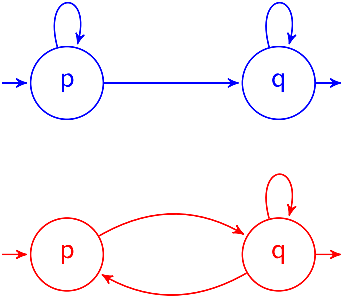
Fibonacci:

$$\sum_{i \in \mathbb{N}} F_i X^i = \frac{X}{1 - X - X^2}$$

The subclass of poly-rational sequences

Theorem: The following classes 1 to 5 are equal.

1. **Polynomially ambiguous weighted automata**
2. Copyless cost register automata
3. Poly-rational expressions
4. LRS with roots of rationals as eigenvalues
5. Rational power series with roots of rationals as poles



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$$\begin{cases} x := x + 5y \\ y := 2z \\ z := 1 \end{cases}$$

$$\begin{cases} x := x \\ y := x + y \end{cases}$$

$$\begin{cases} x := x + y \\ y := 1 \end{cases}$$

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$$\text{Shift}(a, u) = \langle a, u_0, u_1, \ldots \rangle$$

$$\text{Shuffle}(u^1, u^2) = \langle u^1_0, u^2_0, u^1_1, u^2_1, \ldots \rangle$$

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$$\mathrm{PolyRat} = \mathrm{Rat}[\mathrm{Arith} \cup \mathrm{Geo}, +, \times, \mathrm{Shift}, \mathrm{Shuffle}]$$

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$$Q(x) = 0$$

$$\Rightarrow$$

x is a root of rational number

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Identity:

$$Q(x) = X^2 - 2X + 1$$

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5. **Rational power series with roots of rationals as poles**

$$\sum_{i\in\mathbb{N}} u_i X^i = \frac{P}{Q}$$

with P and Q two polynomials and the roots of Q are roots of rational numbers.

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Identity:

$$\sum_{i \in \mathbb{N}} iX^i = \frac{X}{(X-1)^2}$$

Another example:

$$1 + 2X - X^2 - 2X^3 + \cdots = \frac{1+2X}{1+X^2}$$

Separation

Fibonacci $\notin$ P-WA

Complexity

1. The Skolem problem is decidable for our class [Rebiha14].
2. The Skolem problem is NP-hard for our class [Akshay17].
3. It is open if it is in NP.

Thanks!

Sources

1. Rachid Rebiha, Arnaldo Vieira Moura, and Nadir Matringe. On the termination of linear and affine programs over the integers.CoRR, abs/1409.4230, 2014.
2. S. Akshay, Nikhil Balaji, and Nikhil Vyas. Complexity of restricted variants of skolem and related problems. In 42nd International Symposium on Mathematical Foundations of Computer Science, MFCS 2017, August 21-25, 2017 - Aalborg, Denmark, pages 78:1–78:14,2017.

Some other characterisations

$$\mathsf{DetWA} = \cup_{\lambda \in \mathbb{Q}} \mathsf{Rat}[\mathsf{Geo}_\lambda, \mathsf{shift}, \mathsf{shuffle}]$$

$$\mathsf{FinWA} = \mathsf{Rat}[\mathsf{Geo}, +, \mathsf{shift}, \mathsf{shuffle}]$$

$$k\mathsf{WA} = \sum_{i=1,\ldots,k} \mathsf{Rat}[\mathsf{Geo}, \mathsf{shift}, \mathsf{shuffle}]$$

Strictness of the ambiguity hierarchy