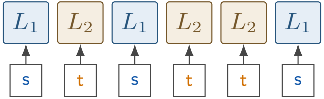
Shuffles of Context-Free Languages along Regular Trajectories

Barloy–Cadilhac–Ockerlund



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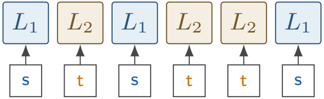

Which regular trajectories preserve context-freeness?

Where we are

Trajectories schedule two words

$$L_1 = \{a^n b^n c^m : n, m \geq 1\}$$

$$L_2 = \{x^m y^n z^n : m, n \geq 1\}$$

from L_1	a	a	b	b	c
from L_2	x	y	y	z	z

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trajectory word c = source schedule

from L_1	a	a	b	b	c					
from L_2	x	y	y	z	z					
trajectory	s	t	s	t	t	s	s	t	s	t

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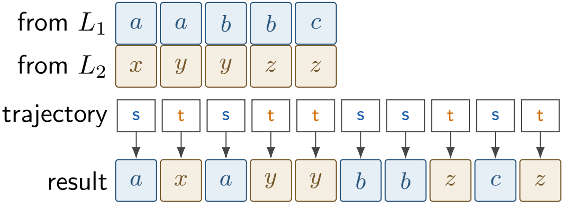

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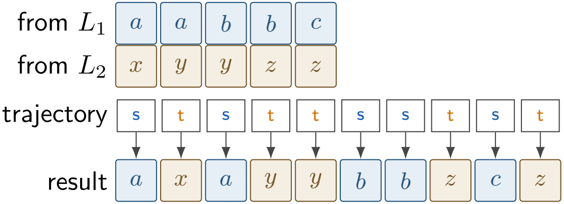
$$u \sqcup\!\sqcup_c v$$



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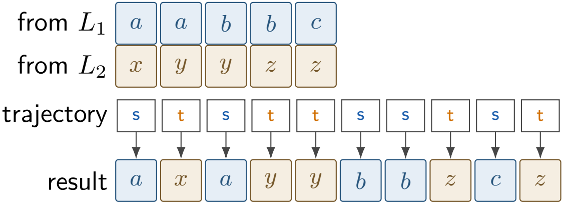


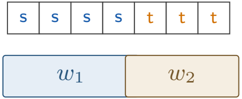
$$u \sqcup\!\sqcup_c v$$

$$L_1 \sqcup\!\sqcup_T L_2 = \{u \sqcup\!\sqcup_c v : \ u \in L_1, \ v \in L_2, \ c \in T\}$$

T regular language over $\{\mathsf{s}, \mathsf{t}\}$

Three trajectories to keep in mind

s^*t^*



concatenation

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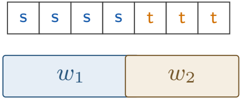

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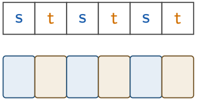
$(st)^*$



strict alternation

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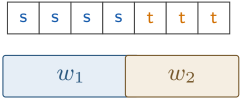

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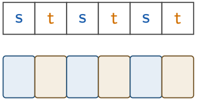
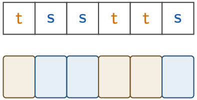


strict alternation

$(s + t)^*$



full shuffle

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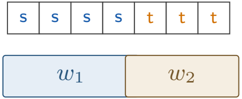

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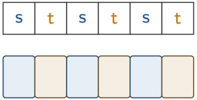

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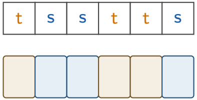
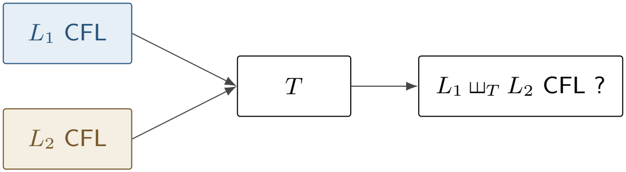


full shuffle

Many other scheduling constraints.

What can go wrong for CFLs?



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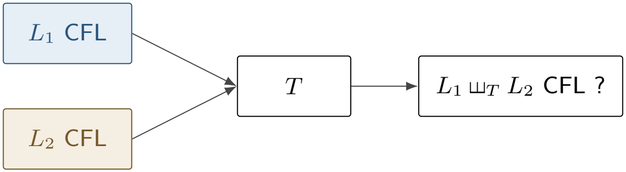

- regular languages: closed under full shuffle

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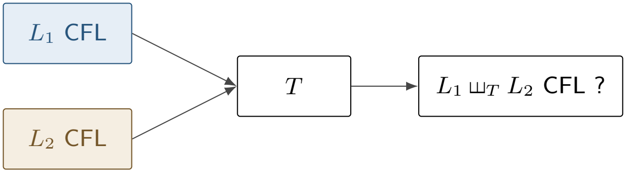

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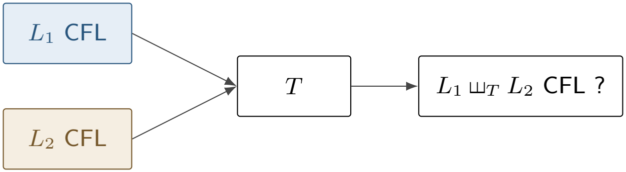

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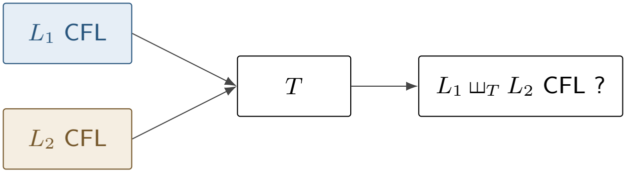

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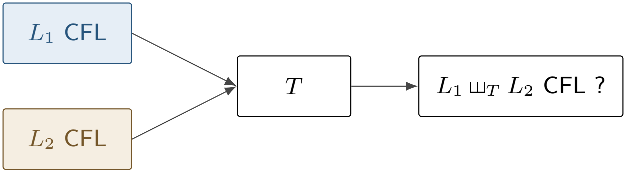

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- test classes without REG: CFL' and DCFL'

- $\mathcal{C}$-safe: all $\mathcal{C}$-pairs stay CFL
- $\mathcal{C}$-hostile: all $\mathcal{C}$-pairs leave CFL
- $\mathcal{C}$-mixed: both behaviors occur

Results

Theorem

Many trajectories including

$$\mathsf{s}^*\mathsf{t}^*\mathsf{s}^*\mathsf{t}^* \qquad \text{and} \qquad (\mathsf{st})^*\mathsf{t}^*(\mathsf{st})^*\mathsf{t}^*$$

are CFL'-hostile.

Results

Theorem

Many trajectories including

$$\mathsf{s}^*\mathsf{t}^*\mathsf{s}^*\mathsf{t}^* \qquad \text{and} \qquad (\mathsf{st})^*\mathsf{t}^*(\mathsf{st})^*\mathsf{t}^*$$

are CFL'-hostile.

Theorem

Given a trajectory T, it is decidable whether it is

- DCFL'-safe $\qquad \mathsf{s}^*\mathsf{t}^*$
- DCFL'-hostile $\qquad (\mathsf{s}+\mathsf{t})^*$
- DCFL'-mixed $\qquad (\mathsf{st})^*$

Where we are

A PDA must match two regions

Investigate PDA runs.

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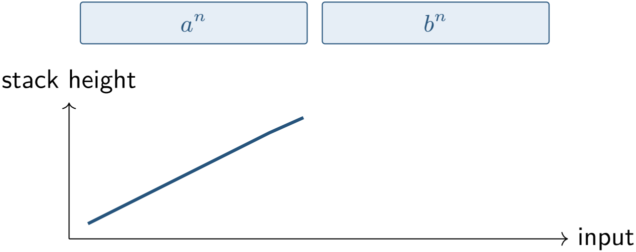

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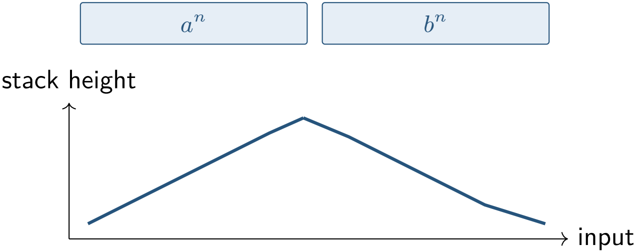

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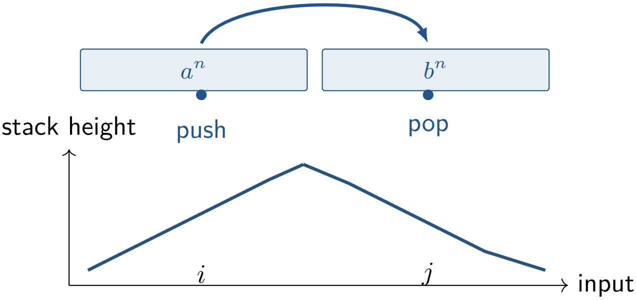

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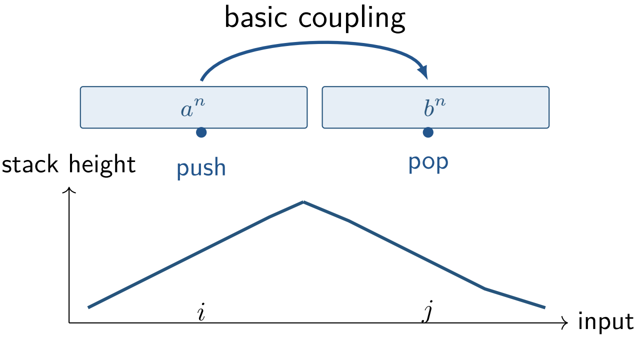
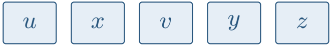


Coupling: element pushed at i is popped at j

Couplings: push in one factor, pop in another



$$w = uxvyz$$

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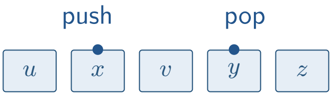

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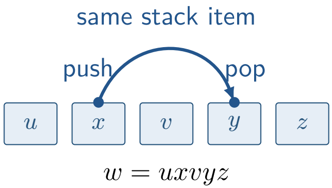

Blocks x and y coupled for the run

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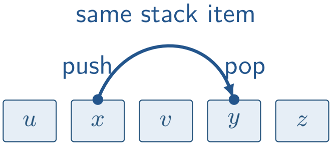

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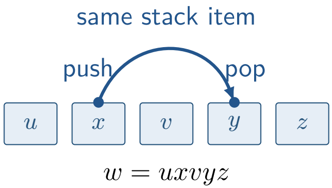

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Definition

Blocks x and y are A-*coupled* iff every accepting run of A on w couples them.

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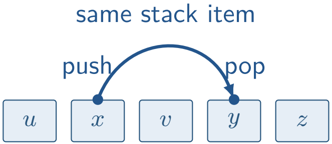

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forced stack dependency

Crossing couplings violate stack discipline

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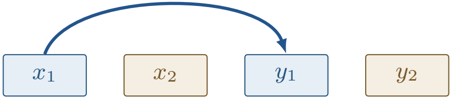

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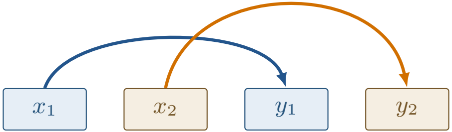

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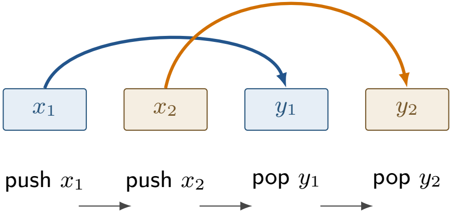
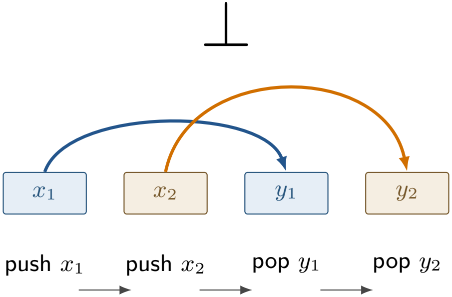


LIFO wants y_2 before y_1

Crossing couplings violate stack discipline



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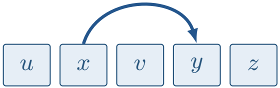
The Coupling Lemmas provide the arcs

Coupling Lemma



- $L \in \mathrm{CFL}'$: forced coupled factors x, y for any size of u, v, z

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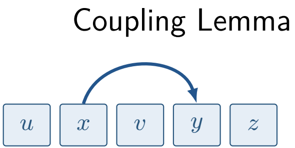

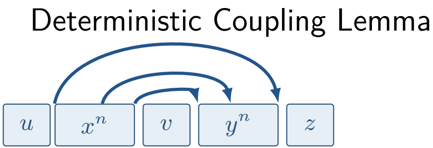

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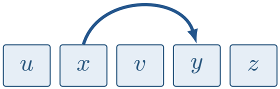
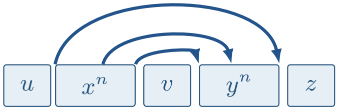


Deterministic Coupling Lemma



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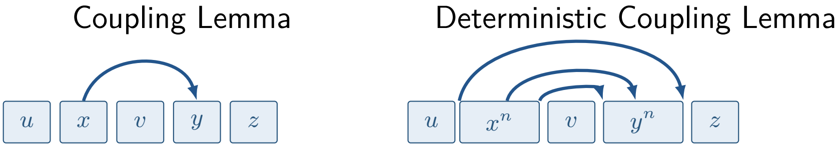

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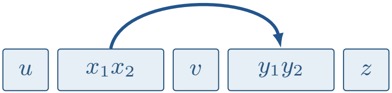
- The PDA is not assumed deterministic!
- trajectory makes these arcs cross

Resilient couplings survive the shuffle

$$u \quad x_1x_2 \quad v \quad y_1y_2 \quad z$$

Resilient couplings survive the shuffle



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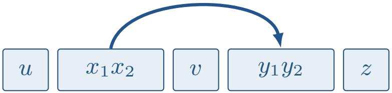
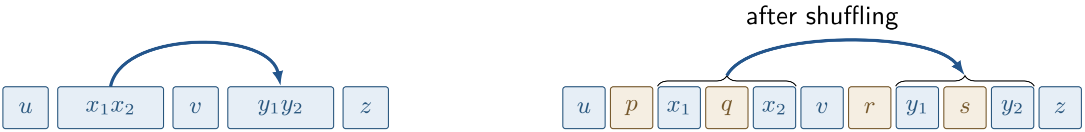


after shuffling

Resilient couplings survive the shuffle



Resilient coupling: there is a coupling after any shuffle

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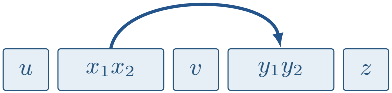

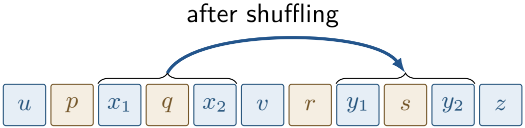

Resilient coupling: there is a coupling after any shuffle

Resilience lemmata: Given a PDA A for $L_1 \sqcup\!\sqcup_T L_2$, the preceding couplings for CFL' and DCFL' can be assumed A-resilient.

Where we are

Example: $s^*t^*s^*t^*$

$$T = s^*t^*s^*t^*$$

s^*	t^*	s^*	t^*

Example: $s^*t^*s^*t^*$

$$T = s^*t^*s^*t^*$$

$$w_1 = x_1 y_1$$

Example: s*t*s*t*

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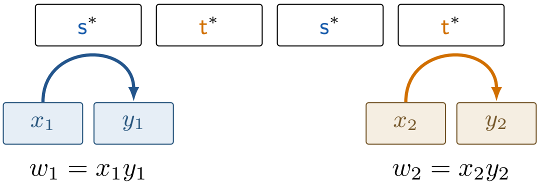

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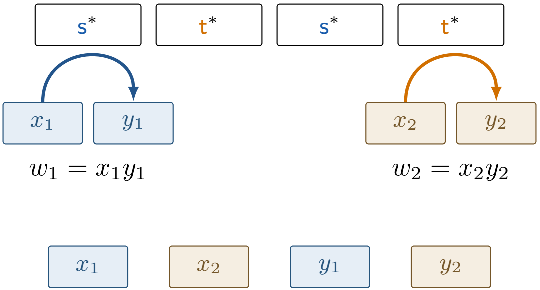

$$w = x_1x_2y_1y_2$$

Example: s*t*s*t*

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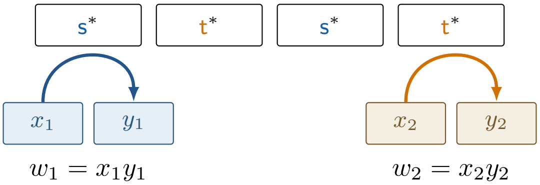

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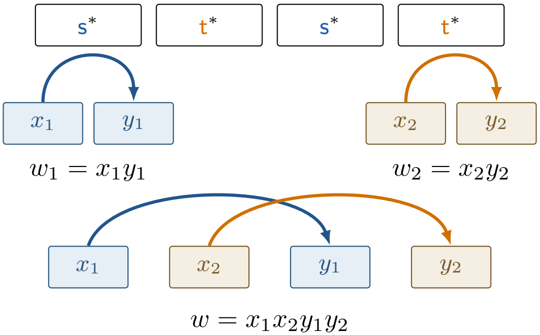

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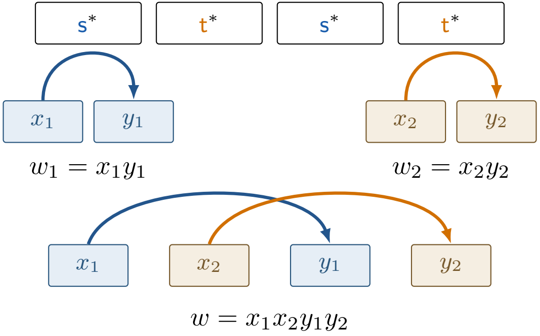

crossing couplings

Example: $(st)^*t^*(st)^*t^*$

$$T = (st)^*t^*(st)^*t^*$$

$(st)^*$	t^*	$(st)^*$	t^*

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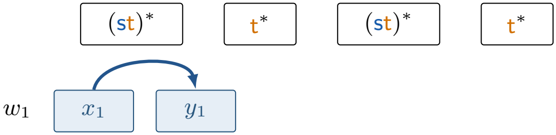

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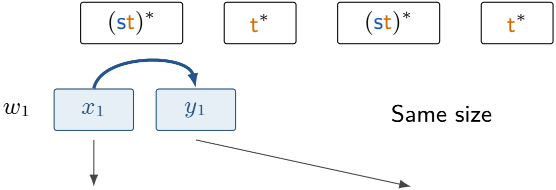

Same size

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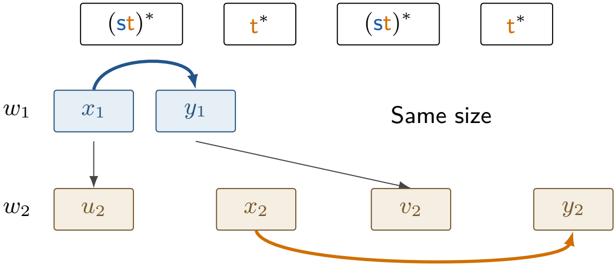
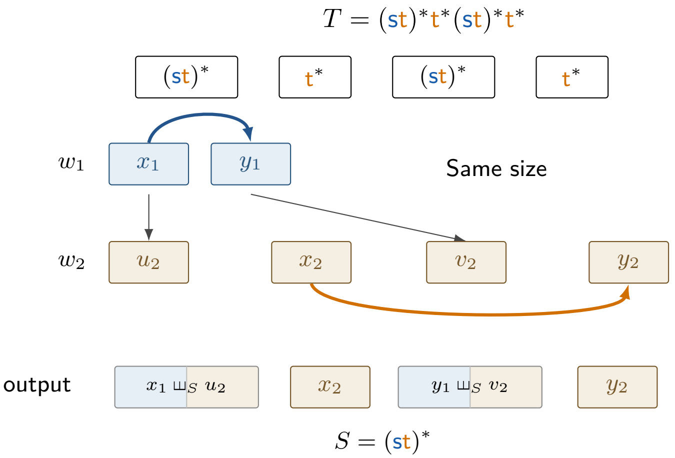


Same size

Example: $(\mathsf{st})^*\mathsf{t}^*(\mathsf{st})^*\mathsf{t}^*$



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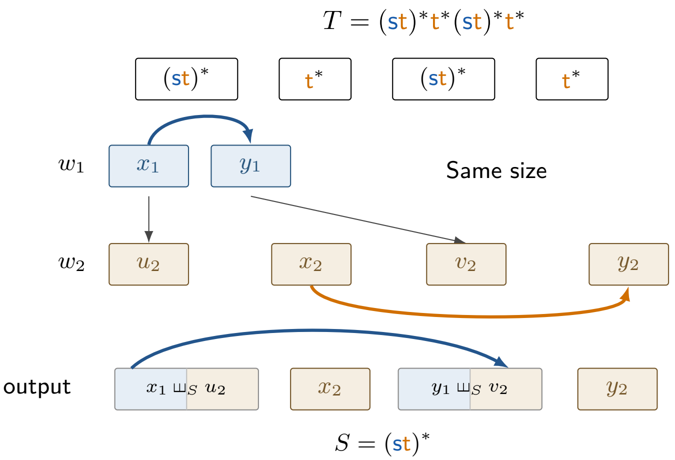

Example: $(\mathsf{st})^*\mathsf{t}^*(\mathsf{st})^*\mathsf{t}^*$

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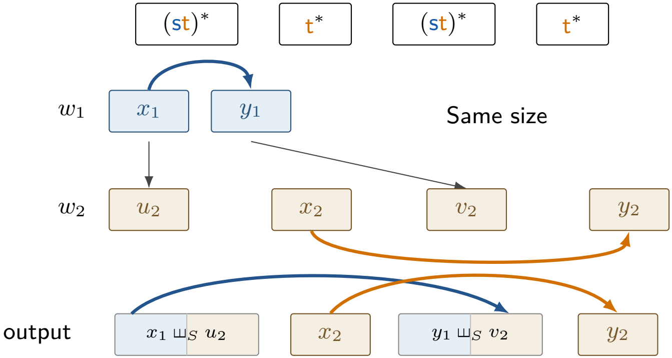

$$S = (\mathsf{st})^*$$

Example: $(st)^*t^*(st)^*t^*$

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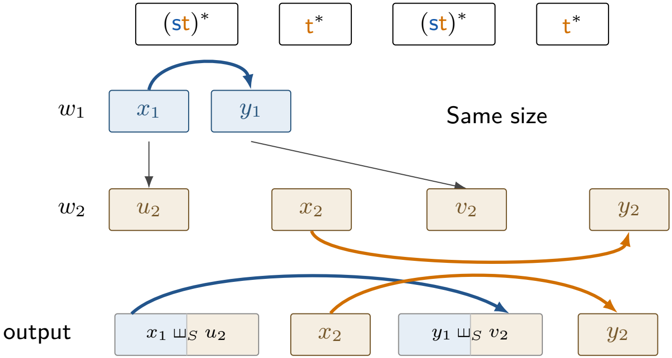

$$S = (st)^*$$

crossing couplings

Formal statement: the first hostile trajectories

Theorem

The trajectories

$$s^*t^*s^*t^* \qquad \text{and} \qquad (st)^*t^*(st)^*t^*$$

are CFL'-hostile.

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Open for CFL' here: $(s^* + t^*)(st)^*(s^* + t^*)$.

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Open for CFL' here: $(\mathsf{s}^* + \mathsf{t}^*)(\mathsf{st})^*(\mathsf{s}^* + \mathsf{t}^*)$.

... but it is DCFL'-hostile!

Where we are

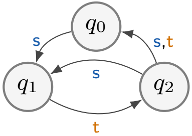
From trajectories to hard SCC-patterns

From trajectories to hard SCC-patterns

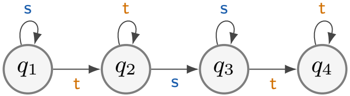
regular trajectory T → DFA for T → SCC patterns

Hard SCCs

one SCC with both letters



Interleaved SCCs



From trajectories to hard SCC-patterns

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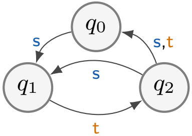

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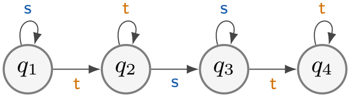

What does $hard(A)$ tell us?

▷ $hard(A) = \emptyset$: CFL'-safe

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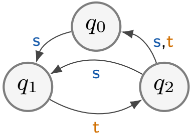

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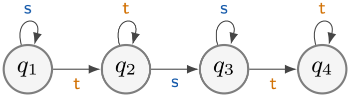

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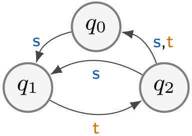

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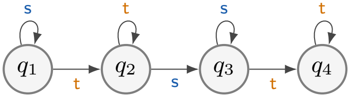

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- ▷ Question: $hard(A) \neq \emptyset$ $\Rightarrow$ DCFL'-hostile?

From trajectories to hard SCC-patterns

regular trajectory T → DFA for T → SCC patterns

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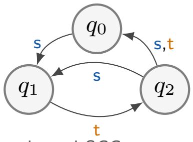
one SCC with both letters



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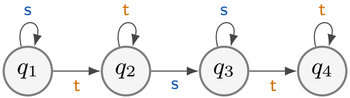

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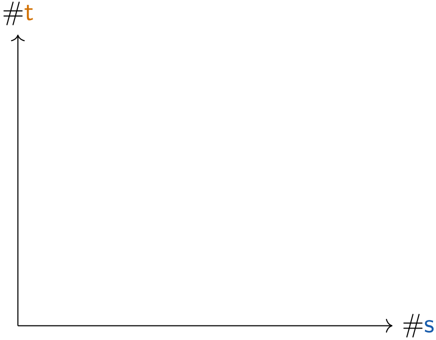
- Question: $hard(A) \neq \emptyset$
  $\Rightarrow \mathrm{DCFL}'$-hostile?
- Not always: length pairs matter

Lengths become points in $\mathbb{N}^2$

What are the lengths of words shufflable with a hard pattern of T?



$$\Psi(c) = \left(|c|_{s},\ |c|_{t}\right)$$

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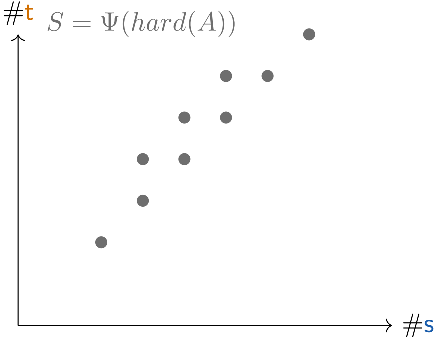

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- S: hard length pairs.

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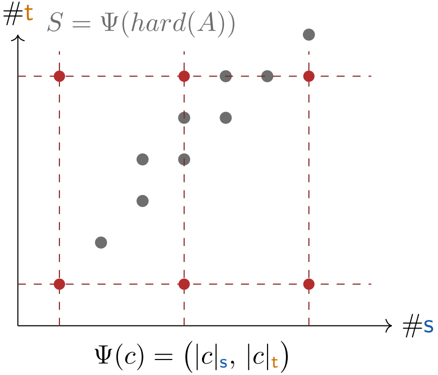

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- grid: $P \times Q$, with P, Q infinite arithmetic progressions.

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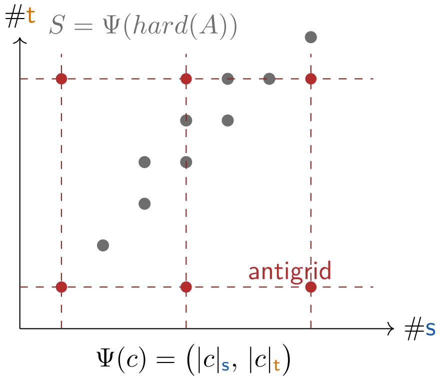

- S: hard length pairs.
- grid: $P \times Q$, with P, Q infinite arithmetic progressions.
- antigrid for S: grid disjoint from S.

The DCFL' trichotomy

DFA A for T; set $S = \Psi(hard(A))$.

case	**example trajectory**	**hard-length geometry**

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DFA A for T; set $S = \Psi(hard(A))$.

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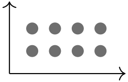
1. no antigrid $\Rightarrow$ hostile	$T = \mathsf{s}^*\mathsf{t}^*\mathsf{s}^*\mathsf{t}^*$	

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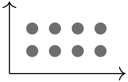
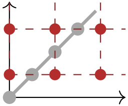
1. no antigrid $\Rightarrow$ hostile	$T = \mathsf{s}^*\mathsf{t}^*\mathsf{s}^*\mathsf{t}^*$	
2. antigrid exists $\Rightarrow$ mixed	$T = (\mathsf{st})^*$	

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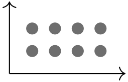
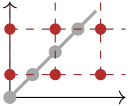
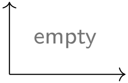
1. no antigrid $\Rightarrow$ hostile	$T = \mathsf{s}^*\mathsf{t}^*\mathsf{s}^*\mathsf{t}^*$	
2. antigrid exists $\Rightarrow$ mixed	$T = (\mathsf{st})^*$	
3. $S = \emptyset$ $\Rightarrow$ safe	$T = \mathsf{s}^*\mathsf{t}^*\mathsf{s}^*$	

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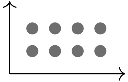
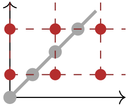
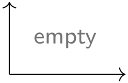
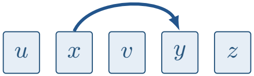
1. no antigrid $\Rightarrow$ hostile	$T = s^*t^*s^*t^*$	
2. antigrid exists $\Rightarrow$ mixed	$T = (st)^*$	
3. $S = \emptyset$ $\Rightarrow$ safe	$T = s^*t^*s^*$	

decidable conditions

Takeaway

▷ Coupling Lemmas provide couplings.



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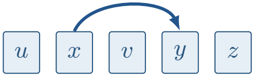
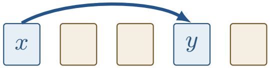


▷ Resilience Lemmas preserve them through shuffles.



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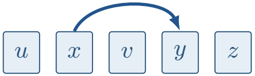

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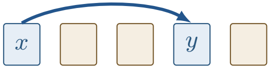

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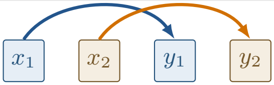

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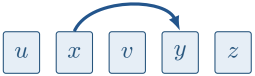

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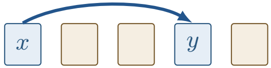

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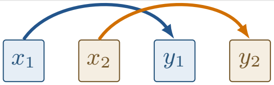

- ▷ DCFL' trichotomy.