The Regular Languages in Circuit Classes

Corentin Barloy

Université de Lille

Joint work with Michaël Cadilhac, Charles Paperman and Thomas Zeume

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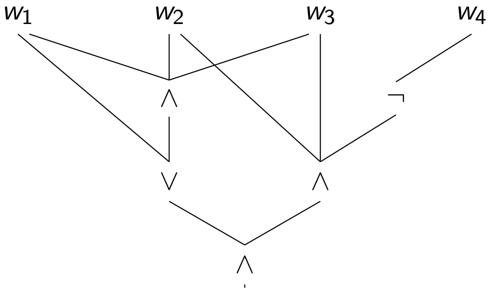
Circuits



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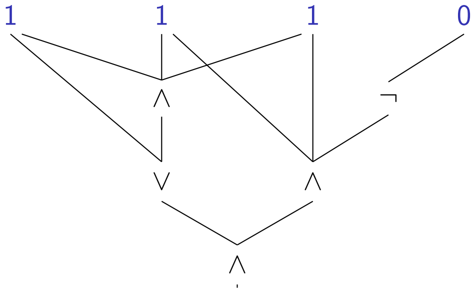

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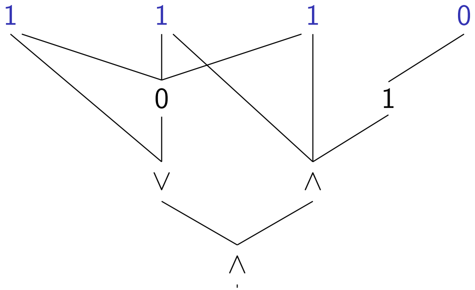

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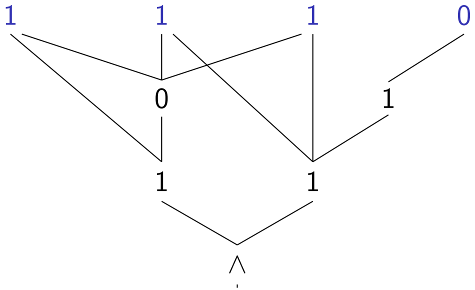

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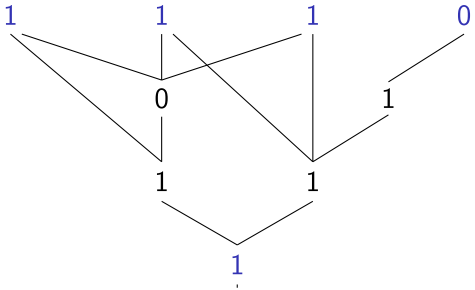

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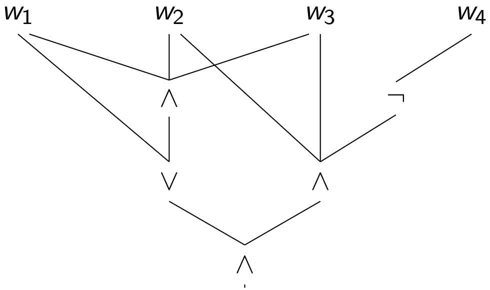
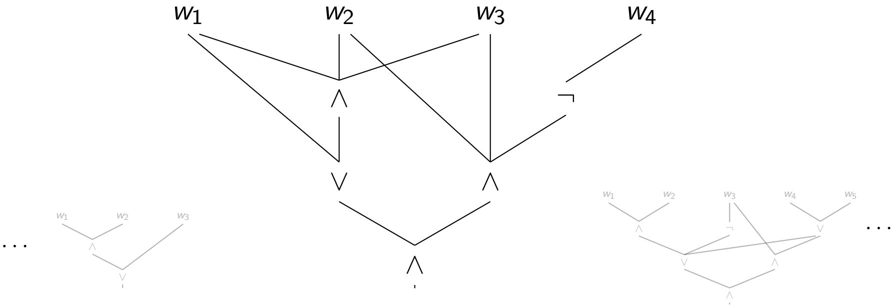


$L_4 = \{1110, \ldots\}$

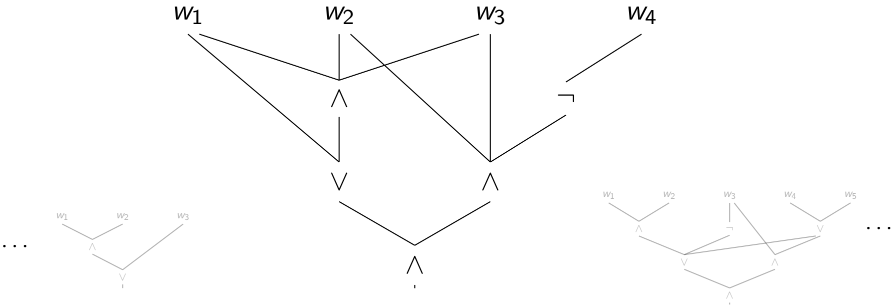
Circuits



$L_4 = \{1110, \dots\}$ $\qquad$ $L = \bigcup_{i \in \mathbb{N}} L_i$

Circuits

size = number of gates



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size $=$ number of gates

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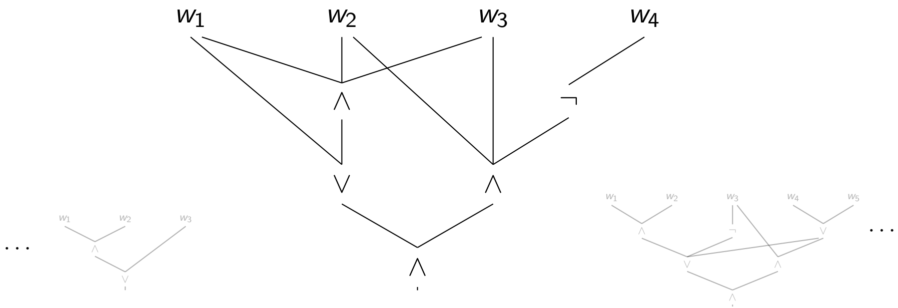

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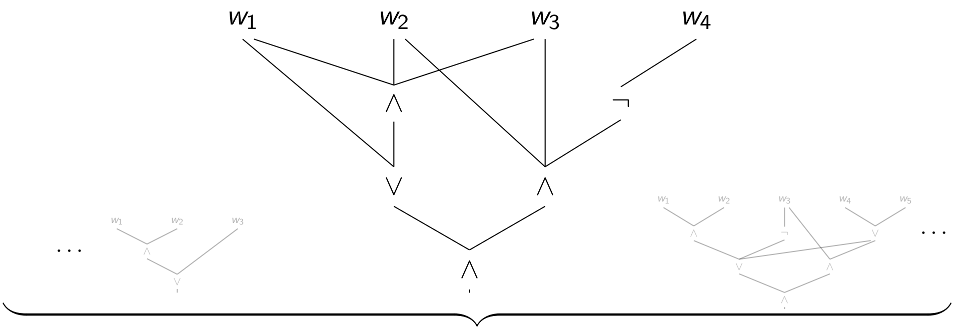

Uniformity = definable in P, LOGSPACE, ...

$L_4 = \{1110, \dots\}$ $\qquad$ $L = \bigcup_{i \in \mathbb{N}} L_i$

Motivations

P vs NP

Motivations

$$\mathsf{P} \subseteq \mathsf{P/poly} \qquad \text{vs} \qquad \mathsf{NP}$$

Motivations

algorithms in P
with polynomial advice

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circuits with
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P-uniform circuits with
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LOGSPACE-uniform circuits with
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P-uniform circuits with
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circuits with
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Shannon: Almost every function requires exponential size.

Motivations

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$$\mathsf{P} \subseteq \mathsf{P/poly} \qquad \text{vs} \qquad \mathsf{NP}$$

P-uniform circuits with
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circuits with
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(Close to hardware)

Shannon: Almost every function requires exponential size.

Classes of small circuits

$$\begin{tikzcd}
 & \mathsf{ACC}^0[\mathcal{P}] \arrow[dr] & & \mathsf{TC}^0 \arrow[dr] & \\
\mathsf{AC}^0 \arrow[ur] & & \mathsf{ACC}^0 \arrow[ur] & & \mathsf{NC}^1
\end{tikzcd}$$

Classes of small circuits

$$\mathsf{AC}^0 \rightarrow \mathsf{ACC}^0[\mathcal{P}] \rightarrow \mathsf{ACC}^0 \rightarrow \mathsf{TC}^0 \rightarrow \mathsf{NC}^1$$

polynomial size
constant depth
unbounded fan-in

Classes of small circuits

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\begin{array}{c}\text{polynomial size}\\ \text{constant depth}\\ \text{unbounded fan-in}\end{array} & & & & \begin{array}{c}\text{polynomial size}\\ \text{logarithmic depth}\\ \text{fan-in } 2\end{array}
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$$

Classes of small circuits

$$\mathsf{AC}^0 \rightarrow \mathsf{ACC}^0[\mathcal{P}] \rightarrow \mathsf{ACC}^0 \rightarrow \mathsf{TC}^0 \rightarrow \mathsf{NC}^1$$

$\mathsf{AC}^0 \leftrightarrow$ polynomial size
constant depth
unbounded fan-in

$\mathsf{NC}^1 \leftrightarrow$ polynomial size
logarithmic depth
fan-in 2

$\boxed{\text{MOD}}$: number of $1 \equiv_m 0$?

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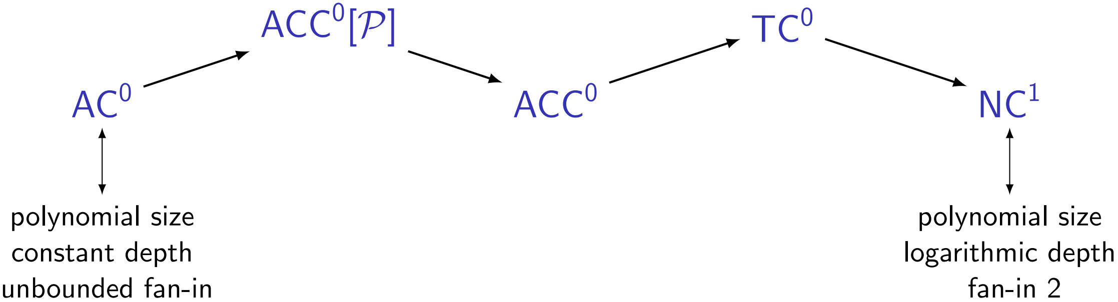

$\boxed{\text{MOD}}$: number of 1 $\equiv_m$ 0 ?

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Classes of small circuits

$$AC^0 \to ACC^0[\mathcal{P}] \to ACC^0 \to TC^0 \to NC^1$$

AC^0 ↔ polynomial size
constant depth
unbounded fan-in

ACC^0 ↔ AC^0 + MOD gates

NC^1 ↔ polynomial size
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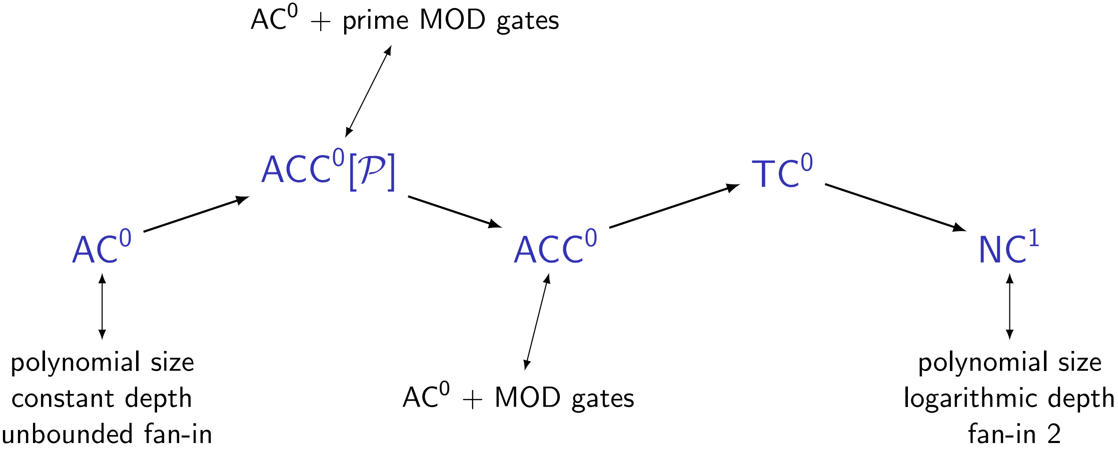

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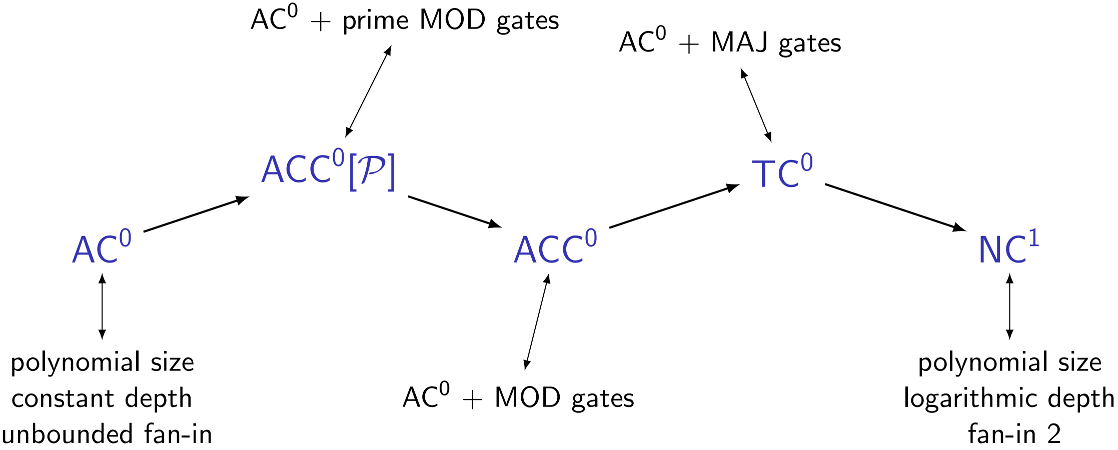

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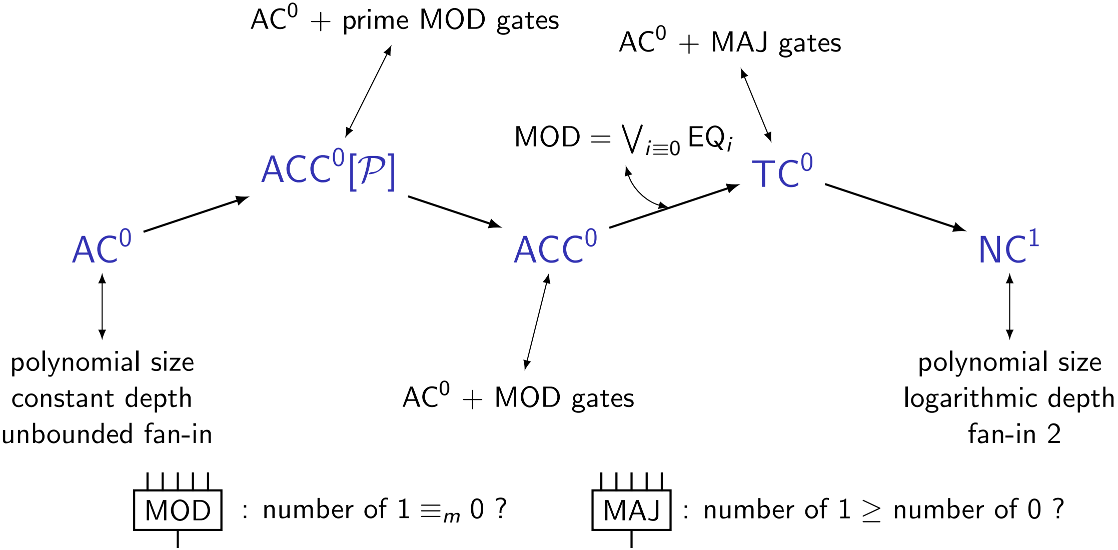

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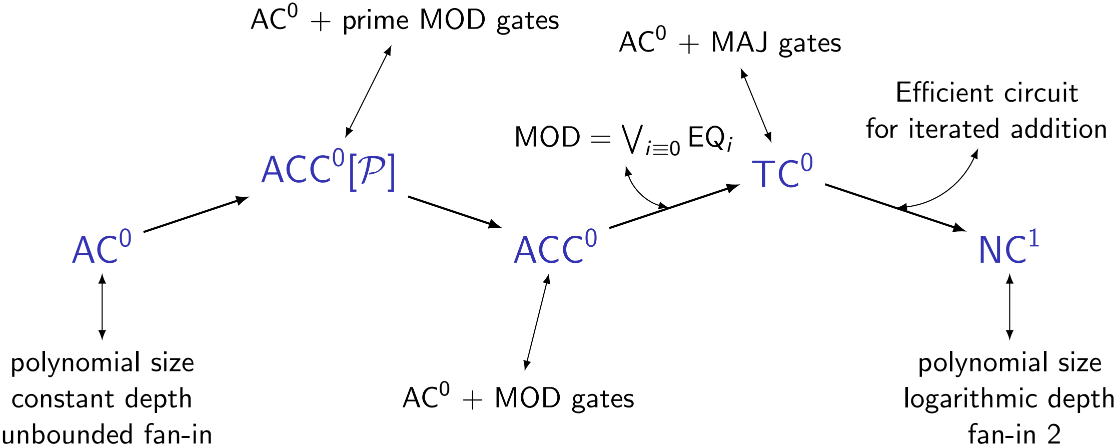

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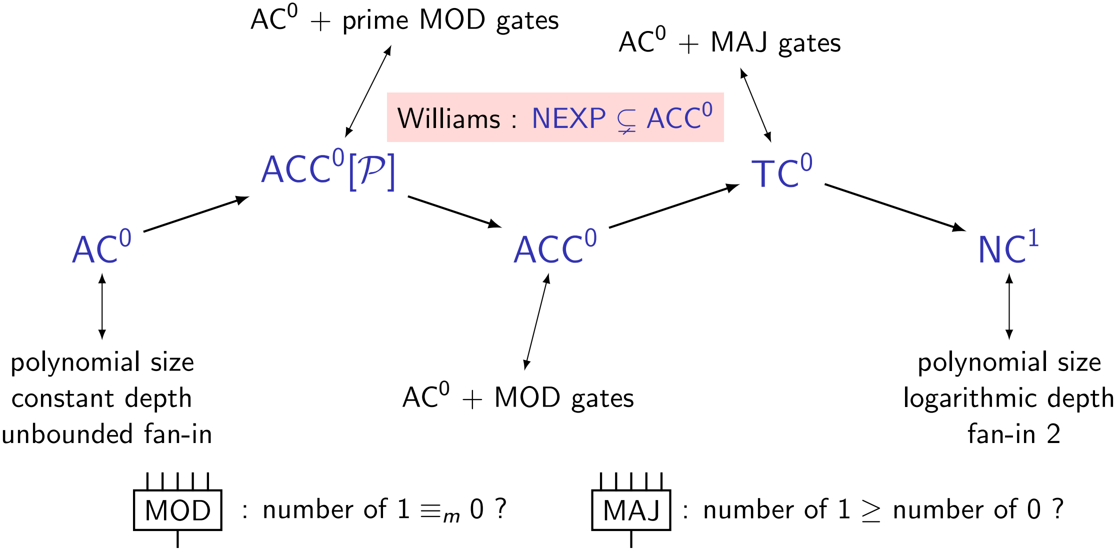

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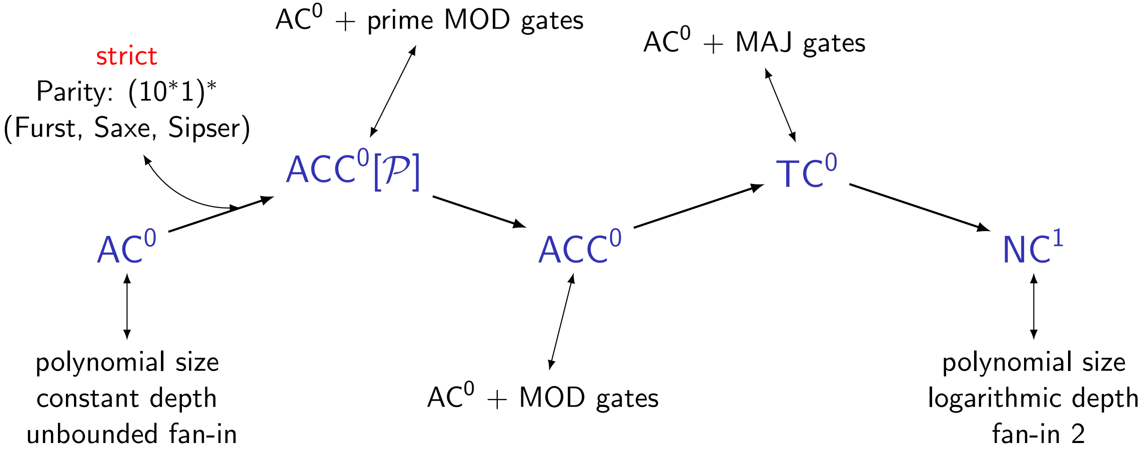

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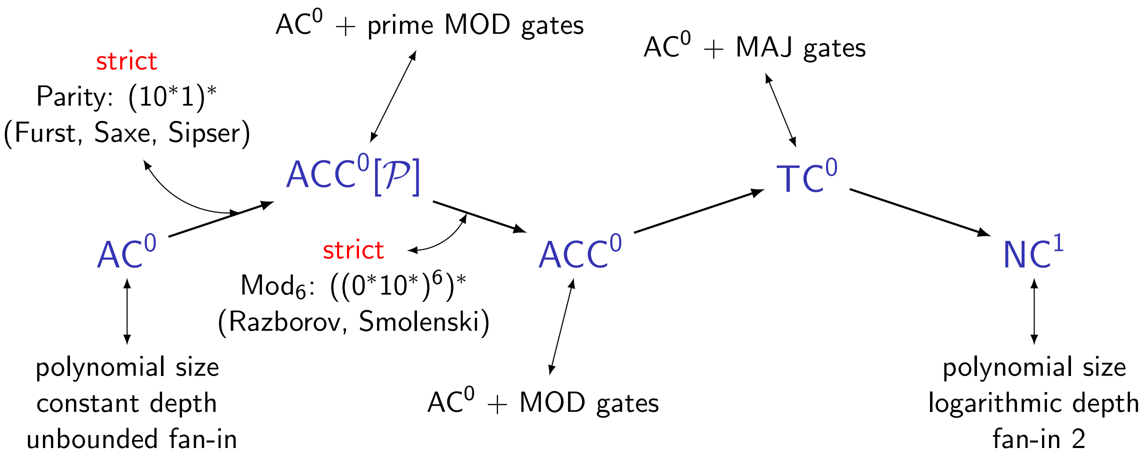

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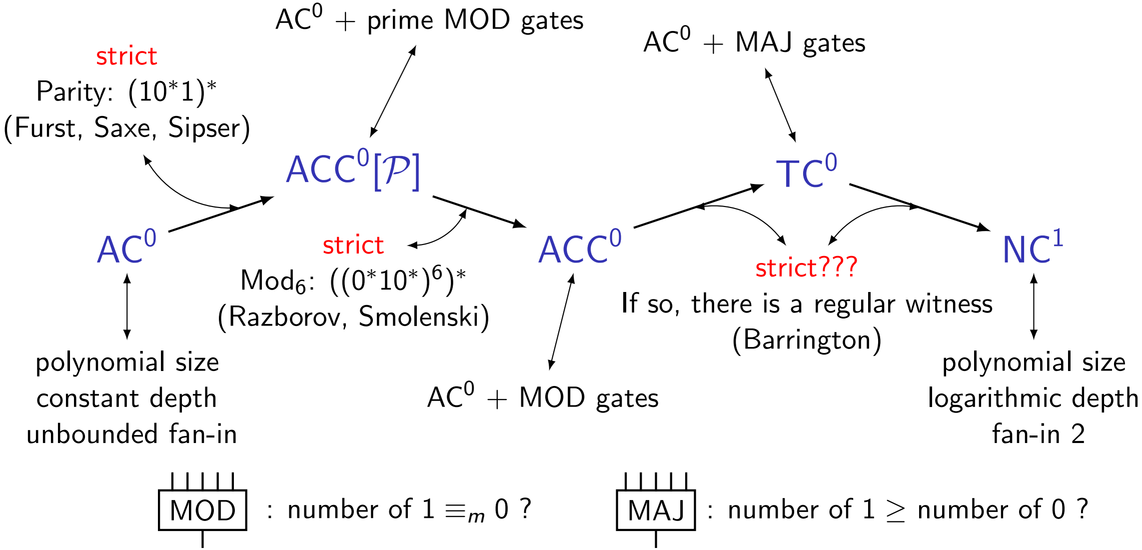

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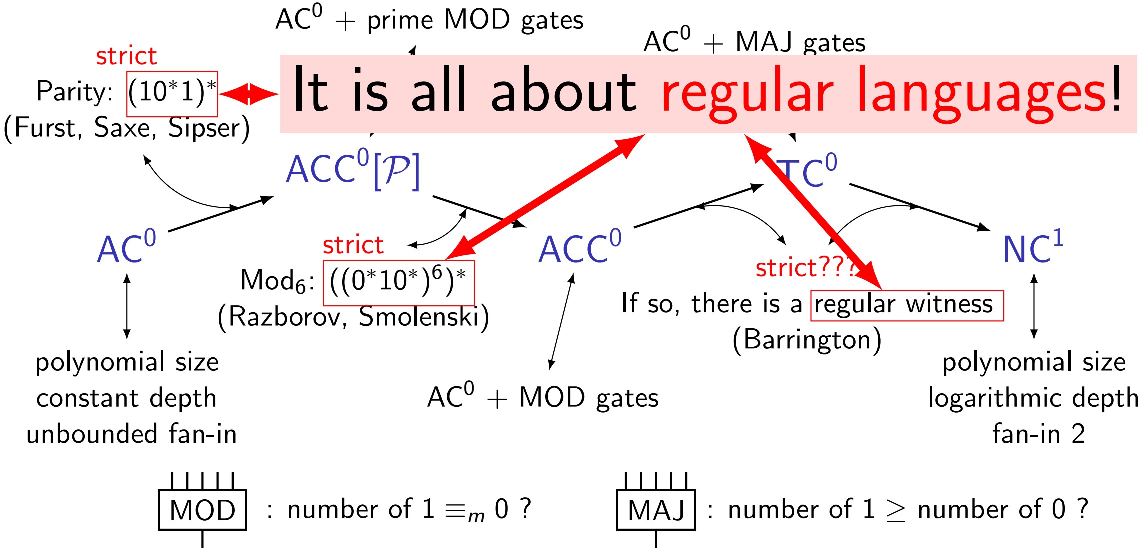

The case of NC^1

Theorem

All regular languages are in NC^1.

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Bounded size circuits for:

- Given a letter a, its transition function
- The composition of two functions
- Whether a function maps the initial state to a final state

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Bounded size circuits for:

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Divide and conquer.

Expressing languages with logic

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Expressiveness

Fragments

Numerical predicates

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$$\exists x, \forall y, a(x) \wedge (y > x \Rightarrow b(y))$$
$$\in \Sigma_2[\mathrm{REG}]$$

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$$\exists x, \forall y, a(x) \wedge (y > x \Rightarrow b(y))$$
$$\in \Sigma_2[\mathrm{REG}]$$

$\forall x,$  $(x) \Rightarrow a(x)$

$$\in \Pi_1[\mathrm{ARB}]$$

Circuits and formulas

$$\begin{tikzcd}
 & \mathsf{ACC}^0[\mathcal{P}] \arrow[dr] & & \mathsf{TC}^0 \arrow[dr] & \\
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\end{tikzcd}$$

Circuits and formulas

$$\mathsf{AC}^0 \to \mathsf{ACC}^0[\mathcal{P}] \to \mathsf{ACC}^0 \to \mathsf{TC}^0 \to \mathsf{NC}^1$$

FO[ARB]

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$\mathrm{AC}^0 \to \mathrm{ACC}^0[\mathcal{P}] \to \mathrm{ACC}^0 \to \mathrm{TC}^0 \to \mathrm{NC}^1$

AC^0: FO[ARB]

$\mathrm{ACC}^0[\mathcal{P}]$: FO+MOD[ARB] (prime)

ACC^0: FO+MOD[ARB]

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 & \text{(prime)} & & & \\
\mathrm{FO}[\mathrm{ARB}] & & \mathrm{FO{+}MOD}[\mathrm{ARB}] & &
\end{tikzcd}
$$

$\exists^{mod}_{q,r}$ = # satisfiyng assignements is $\equiv$ r mod q

$\exists^{maj}$ = a majority of the assignements are satisfying

Central question

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What are the regular languages of $\mathcal{L}[ARB]$ *?*

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$\mathrm{ACC}^0[\mathcal{P}]$

False

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Open

- FO with composite modular quantifiers
- FO with two variables
- Σ_k, $k \geq 3$

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Circuits with
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(AND $\in \mathrm{CC}^0_6$?)

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Linear AC^0
(complexity of addition)

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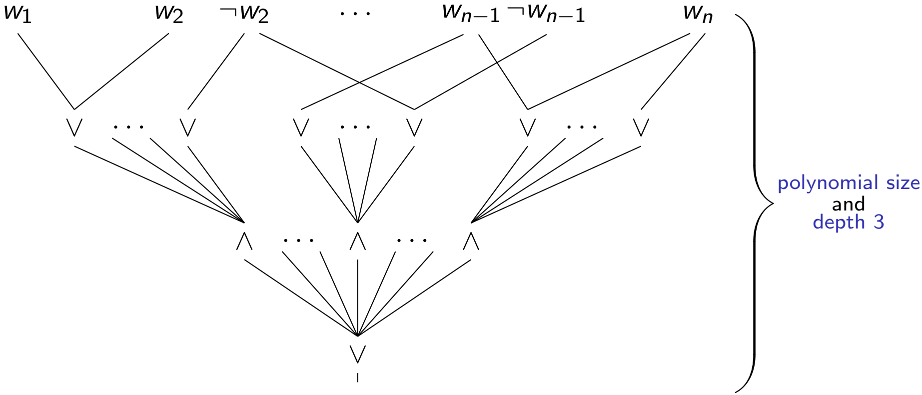
$\mathrm{ACC}^0[\mathcal{P}]$

Our result: $\Sigma_2[\mathrm{ARB}] \cap \mathrm{Reg} = \Sigma_2[\mathrm{REG}]$

Linear AC^0 (complexity of addition)

The circuit class Σ_2

$w = w_1 w_2 \cdots w_{n-1} w_n$



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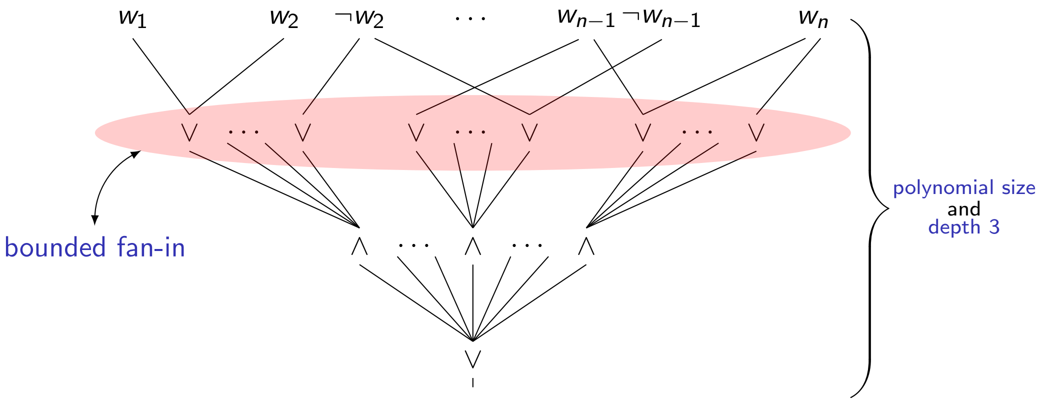

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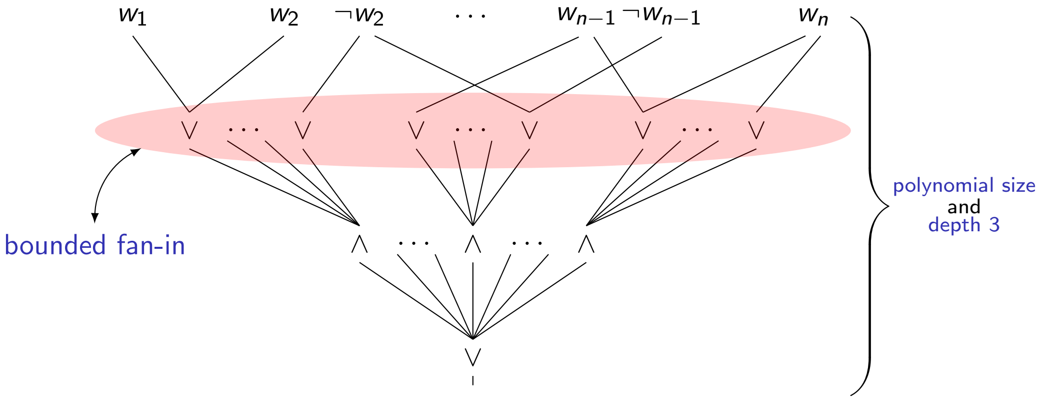

It is equivalent to Σ_2[ARB].

The case of Σ_2 (B,Cadilhac, Paperman, Zeume)

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- $\supseteq$: Immediate.
- $\subseteq$: Take a regular language not in $\Sigma_2[\mathrm{REG}]$, show that it is not in $\Sigma_2[\mathrm{ARB}]$.

The case of Σ_2 (B,Cadilhac, Paperman, Zeume)

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- $\supseteq$: Immediate.
- $\subseteq$: Take $\underbrace{\text{a regular language not in } \Sigma_2[\mathrm{REG}]}_{\text{Algebra}}$, show that $\underbrace{\text{it is not in } \Sigma_2[\mathrm{ARB}]}_{\text{Circuit lower bound}}$.

Proof sketch

$\boxed{\text{Algebra}}$ $\boxed{\text{Lower bound}}$

Proof sketch

Theorem (Pin, Weil)

$\mathcal{L}$ in Σ_2[REG] iff:

Proof sketch

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$\mathcal{L}$ in Σ_2[REG] iff:

$\forall uxv \in \mathcal{L}$

$$u \quad x \quad v \in \mathcal{L}$$

Proof sketch

Theorem (Pin, Weil)

$\mathcal{L}$ in Σ_2[REG] iff:

$\forall uxv \in \mathcal{L}$ such that x can be iterated

$$u \quad xxxxx \quad v \in \mathcal{L}$$

Proof sketch

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Theorem (Pin, Weil)

$\mathcal{L}$ in Σ_2[REG] iff:

$\forall uxv \in \mathcal{L}$ such that x can be iterated , then $uxyxv$ is also in $\mathcal{L}$ for every y with the same letters as x.

$$u \quad xyx \quad v \in \mathcal{L}$$

Proof sketch

Definition (limit (Sipser))

Let A be a set of words in $\mathcal{L}$.
A limit for A is a word u:

- *not in $\mathcal{L}$*
- *that can fool every $\vee$ of bounded fan-in that accepts (at least) A.*

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We give here a new method of finding limits, $\underbrace{\text{specially tailored for } \Sigma_2}_{\text{of the form } uxyxv}$.

Conclusion

Also in the paper:

- Straubing's conjecture for Δ_2.

Not in the paper:

- The proof in its full generality.

Future work:

- Go higher in the hierarchy: $\mathcal{B}\Sigma_2$, Σ_3, ...
- Tackle different kind of fragments, like FO_2.