The Regular Languages of First-Order Logic with One Alternation

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Expressing languages with logic

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Numerical predicates

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$$\exists x, \forall y, a(x) \wedge (y > x \Rightarrow b(y)) \in \Sigma_2[\text{REG}]$$

$$\forall x, (x \text{ encodes a cat}) \Rightarrow a(x) \in \Pi_1[\text{ARB}]$$

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FO		FO with two variables
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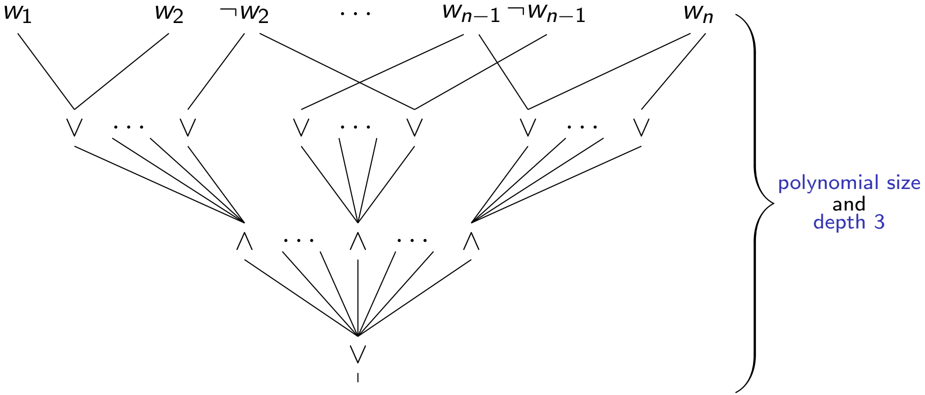
Our result: $\Sigma_2[\mathrm{ARB}] \cap \mathrm{Reg} = \Sigma_2[\mathrm{REG}]$

Linear AC^0
(complexity of addition)

The circuit class Σ_2

$w = w_1 w_2 \cdots w_{n-1} w_n$



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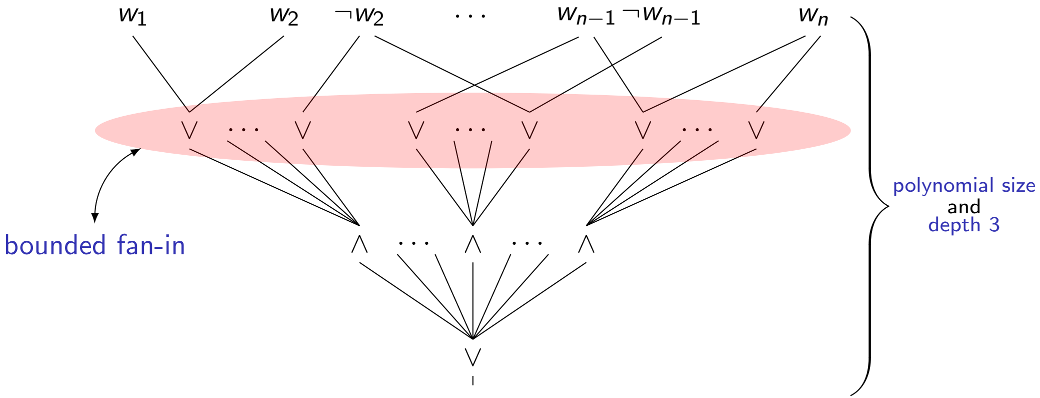

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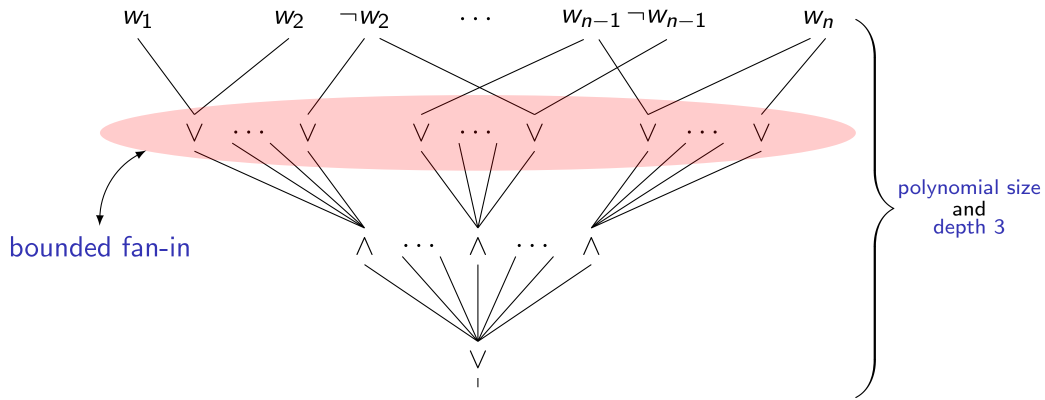

It is equivalent to Σ_2[ARB].

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Algebra | Lower bound

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Theorem (Pin, Weil)

$\mathcal{L}$ in Σ_2[REG] iff:

$\forall uxv \in \mathcal{L}$ such that x can be iterated , then $uxyxv$ is also in $\mathcal{L}$ for every y with the same letters as x.

$$u \quad xyx \quad v \in \mathcal{L}$$

Proof sketch

Definition (limit (Sipser))

Let A be a set of words in $\mathcal{L}$.
A limit for A is a word u:

- *not in $\mathcal{L}$*
- *that can fool every $\vee$ of bounded fan-in that accepts (at least) A.*

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We give here a new method of finding limits, $\underbrace{\text{specially tailored for } \Sigma_2}_{\text{of the form } uxyxv}$.

Conclusion

Also in the paper:

- Straubing's conjecture for Δ_2.

Not in the paper:

- The proof in its full generality.

Future work:

- Go higher in the hierarchy: $\mathcal{B}\Sigma_2$, Σ_3, ...
- Tackle different kind of fragments, like FO_2.