THE REGULAR LANGUAGES OF FIRST-ORDER LOGIC WITH ONE ALTERNATION

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Expressing languages with logic

The set of words of the form $a^n b^n$ can be defined with:

$$\exists x, (\forall y, y \leq x \Rightarrow a(y)) \wedge (\forall y, y > x \Rightarrow b(y)) \wedge (\exists z, \mathrm{End}(z) \wedge \mathrm{Half}(x, z))$$

The expressivity of a logic can depends on:

Possible quantification allowed:

- The full first-order logic: FO,
- Resticting to two variables: FO_2,
- Adding modular quantifiers: FO + MOD,
- Bounded alternation of quantifiers: Σ_k,
- Many more...

Possible numerical predicates:

$$\left.\begin{array}{l}\bullet \text{ The order: } <, \\ \bullet \text{ The regular predicates: } +1, \\ \bullet \text{ The modular predicates: MOD,}\end{array}\right\} \text{The regular predicates: REG.}$$

- Any arbitrary predicates: ARB,
- Many more...

Straubing's central conjecture

Let $\mathcal{L}$ be a logic,

The regular languages.

$$\mathcal{L}[\mathrm{ARB}] \cap \mathrm{Reg} = \mathcal{L}[\mathrm{REG}] \ ?$$

For which logics $\mathcal{L}$ does the regular languages in $\mathcal{L}$[ARB] are exactly the languages of $\mathcal{L}$[REG]?

True	False	Open
• Σ_1, • FO, • FO + MOD(p^k), for a fixed prime p.	• (the exotic) FO + S_5.	• FO + MOD, • FO_2, • Σ_k.

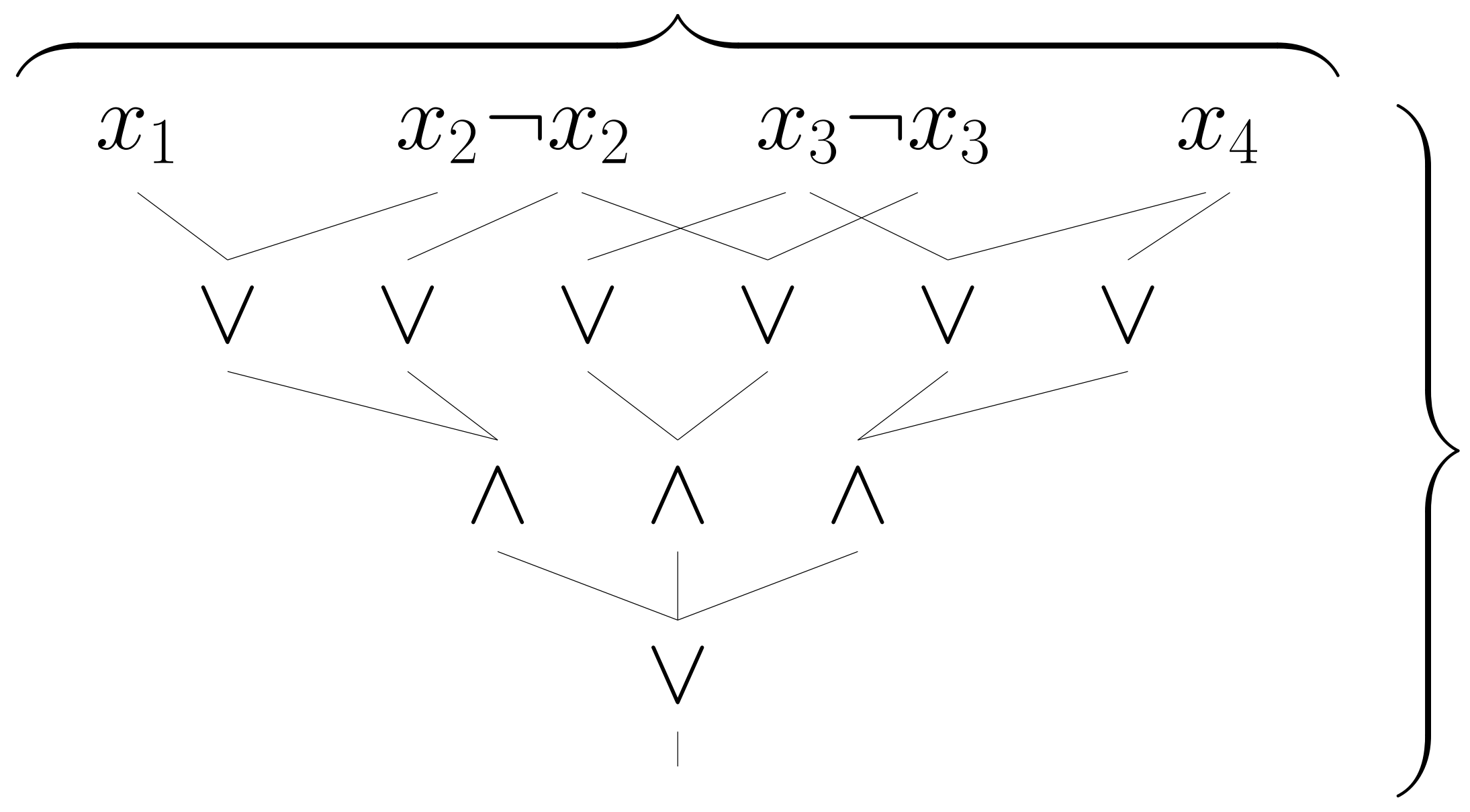
The logic $\Sigma_2[ARB]$

$$\underbrace{\exists x_1, \ldots, x_k}_{\text{A block of existential quantifiers}} \ \underbrace{\forall y_1, \ldots, y_k}_{\text{A block of universal quantifiers}} \ \underbrace{\varphi(x_1, \ldots, y_k)}_{\text{A formula that can use arbitrary predicates}}$$

$$\Longleftrightarrow$$

The circuit class $\exists\forall\exists$

the fan-in of the first level is bounded



polynomial size
depth 3

The main result (with Cadilhac, Paperman and Zeume)

$$\Sigma_2[\mathrm{ARB}] \cap \mathrm{Reg} \cap \mathrm{Neut} = \Sigma_2[\mathrm{REG}] \cap \mathrm{Neut} \ .$$

The class of languages with a neutral letter: a mild technical assumption.

The two parts of the proof:

- An algebraic characterisation of $\Sigma_2[<]$ (Pin and Weil).
- Lower bounds against $\exists\forall\exists$.

A corollary of the proof

The logic Π_2 is defined as Σ_2 but with an initial block of universal quantifiers.
The logic Δ_2 is defined as the class of formulas that can be written both as a Σ_2 formula and as a Π_2 formula.

$$\Delta_2[\mathrm{ARB}] \cap \mathrm{Reg} = \Delta_2[\mathrm{REG}] \ .$$

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